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AN INEQUALITY FOR TOTALLY REAL SURFACES IN COMPLEX SPACE FORMS

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Abstract. For a totally real surface M of a complex space form $\widetilde{M}(4c)$ of arbitrary codimension, we obtain an inequality relating the squared mean curvature $\|H\|^2$, the holomorphic sectional curvature c, the Gauss curvature K and the elliptic curvature K^E of the surface. Using the notion of ellipse of curvature, we obtain a characterization of the equality. An example of a Lagrangian surface of $\mathbf{C}^2$ satisfying the equality case is given.

INTRODUCTION

Let $(\widetilde{M}, g)$ be an n-dimensional Hermitian manifold and M an m-dimensional submanifold of $\widetilde{M}$, endowed with the induced metric. The submanifold M is called *totally real* if for any point p of M and any vector X in T_pM, JX is a normal vector (where J is the almost complex structure on $\widetilde{M}$). Also, a totally real submanifold M is called *Lagrangian* if its dimension is maximal, i.e., $\dim M = \dim_{\mathbf{C}} \widetilde{M}$.

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We denote by $\widetilde{R}$, R and $R^{\perp}$ the curvature tensors of M, $\widetilde{M}$ and the normal curvature tensor, respectively, and by h the second fundamental form of M in $\widetilde{M}$. The well-known equations of Gauss and Ricci are given by:

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \\ + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \tag{0.1}$$

$$\widetilde{R}(X,Y,\xi,\eta) = R^{\perp}(X,Y,\xi,\eta) + g([A_{\xi},A_{\eta}]X,Y), \tag{0.2}$$

for any vector fields X,Y,Z,W tangent to M and ξ,η normal to M.

For a point $p \in M$, let $\{e_1,...,e_m\}$ be an orthonormal basis of the tangent space T_pM and $\{e_{m+1},...,e_{2n}\}$ an orthonormal basis of the normal space $T_p^{\perp}M$. We will use the following standard notations

$$h_{ij} = h(e_i,e_j), \quad h_{ij}^k = g(h(e_i,e_j),e_k),$$

for $i,j \in \{1,...,m\}, k \in \{m+1,...,2n\}$, and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)).$$

The mean curvature vector $H = \frac{1}{m}$ trace h takes the form $H = \frac{1}{m}\sum_{i=1}^{m} h_{ii}$.

1. THE MAIN RESULT

Let M be a totally real surface of the complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ and of complex dimension n. Then the curvature tensor $\widetilde{R}$ is given by

$$\widetilde{R}(X,Y,Z,W) = c[g(X,Z)g(Y,W) - g(X,W)g(Y,Z) +$$

$$+ g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) + 2g(X,JY)g(Z,JW)].$$

For a point $p \in M$, let $\{e_1,e_2\}$ be an orthonormal basis of the tangent plane T_pM and $\{e_3,...,e_{2n}\}$ an orthonormal basis of the normal space $T_p^{\perp}M$.

The *ellipse of curvature* at $p \in M$ is the subspace E_p of the normal space given by

$$E_p = \{h_p(X, X) \mid X \in T_pM, \|X\| = 1\}.$$

For any vector $X = (\cos\theta)e_1 + (\sin\theta)e_2$, $\theta \in [0, 2\pi]$, we have

$$h_p(X, X) = H(p) + (\cos 2\theta)\frac{h_{11} - h_{22}}{2} + (\sin 2\theta)h_{12}.$$

We recall the following result.

Proposition [2]. *If the ellipse of curvature is non-degenerated, then the vectors* $h_{11} - h_{22}$ *and* h_{12} *are linearly independent.*

Using a similar method with [2] and [3] and the above Proposition, we can define a 2-plane subbundle P of the normal bundle, with the induced connection.

We will define then the *elliptic curvature* by the formula

$$K^E = g([A_{e_3}, A_{e_4}]e_1, e_2),$$

where $\{e_1, e_2\}$, $\{e_3, e_4\}$ are orthonormal basis of T_pM and P_p and A is the shape operator.

Remark. This definition of the elliptic curvature coincides with the definition of the *normal curvature* (given by Wintgen [W] and also Guadalupe and Rodriguez [2] by the formula $K^N = g(R^\perp(e_1, e_2)e_3, e_4)$), if the ambient space $\tilde{M}(c)$ is a real space form.

We will prove the following

Theorem. *Let* M *be a totally real surface of the complex space form* $\widetilde{M}(4c)$ *of constant holomorphic sectional curvature* $4c$ *and of complex dimension* n. *Then, at any point* $p \in M$ *we have*

$$\|H\|^2 \geq K - K^E - c.$$

Moreover, the equality sign holds if and only if the ellipse of curvature is a circle.

Proof. We can choose $\{e_1, e_2\}$ such that the vectors $u = \frac{h_{11}-h_{22}}{2}$ and $v = h_{12}$ are perpendicular, in which case they coincide with the half-axis of the ellipse. Then we will take $e_3 = \frac{u}{\|u\|}$ and $e_4 = \frac{v}{\|v\|}$.

From the equation of Ricci (0.2) and the definition of K^E, we have

$$K^E = -\|h_{11} - h_{22}\| \cdot \|h_{12}\| . \tag{1.1}$$

Also, from the Gauss equation (0.1) we obtain the formula of the Gauss curvature K of the totally real surface M of the complex space form $\widetilde{M}(4c)$

$$K = g(h_{11}, h_{22}) - \|h_{12}\|^2 + c. \tag{1.2}$$

By the definition of the mean curvature vector, (1.2) and the relation $\|h\|^2 = \|h_{11}\|^2 + \|h_{22}\|^2 + 2\|h_{12}\|^2$, we have

$$4\|H\|^2 = \|h\|^2 + 2(K - c). \tag{1.3}$$

Then

$$0 \leq (\|h_{11} - h_{22}\| - 2\|h_{12}\|)^2 = \|h\|^2 - 2(K - c) + 4K^E = \tag{1.4}$$

$$= 4\|H\|^2 - 4(K - c) + 4K^E,$$

which is equivalent to

$$\|H\|^2 \geq K - K^E - c. \tag{1.5}$$

The equality sign holds if and only if $\|h_{11} - h_{22}\| = 2\|h_{12}\|$, i.e. $\|u\| = \|v\|$, so the ellipse of curvature is a circle.

2. EXAMPLE

In this section we will give one example of a Lagrangian surface in $\mathbf{C}^2$, endowed with the standard almost complex structure J_0, for which the equality sign holds identically (which we call an *ideal surface*).

Let M be the *rotation surface* of Vrănceanu [7], given by

$$X(u,v) = r(u)(\cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v),$$

where r is a positive C^∞-differentiable function.

Let $\{e_1, e_2\}$ be an orthonormal basis of the tangent plane and $\{e_3, e_4\}$ an orthonormal basis of the normal plane.

Then it is easy to find the following expressions for $e_i, i \in \{1,2,3,4\}$ (see also [6]):

$$e_1 = (-\cos u \sin v, \cos u \cos v, -\sin u \sin v, \sin u \cos v),$$

$$e_2 = \frac{1}{A}(B \cos v, B \sin v, C \cos v, C \sin v),$$

$$e_3 = \frac{1}{A}(-C \cos v, -C \sin v, B \cos v, B \sin v),$$

$$e_4 = (-\sin u \sin v, \sin u \cos v, \cos u \sin v, -\cos u \cos v),$$

where $A = [r^2 + (r')^2]^{\frac{1}{2}}$, $B = r' \cos u - r \sin u$, $C = r' \sin u + r \cos u$.

Also, after technical calculations, we find

$$h^3_{11} = \frac{1}{[r^2 + (r')^2]^{\frac{1}{2}}}, \quad h^3_{12} = 0, \quad h^3_{22} = \frac{-rr'' + 2(r')^2 + r^2}{[r^2 + (r')^2]^{\frac{3}{2}}},$$

$$h^4_{11} = 0, \quad h^4_{12} = -\frac{1}{[r^2 + (r')^2]^{\frac{1}{2}}}, \quad h^4_{22} = 0.$$

It is easily seen that M is a totally real surface of maximum dimension, so is a *Lagrangian surface* of $\mathbf{C}^2$. Moreover, M satisfies the equality sign of the inequality proved above (it is an *ideal surface*) if and only if

$$r(u) = \frac{1}{(|\cos 2u|)^{\frac{1}{2}}}$$

(the ellipse of curvature at every point of M is a circle).

In this case, M is a minimal surface (see [5]) and $X = c_1 \otimes c_2$ is the tensor product immersion of $c_1(u) = \frac{1}{(|\cos 2u|)^{\frac{1}{2}}}(\cos u, \sin u)$ (an orthogonal hyperbola) and $c_2(v) = (\cos v, \sin v)$ (a circle).

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