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BANACH PRECOMPACT ELEMENTS OF A LOCALLY $M-$CONVEX B_O-ALGEBRA

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Abstract. In this paper, we present and prove that Banach precompactness of an element $y \in A$ of a locally $m-$convex B_o- algebra is inherited by the elements of $A(y)$.

Notations and definitions. Let A be an algebra over the complex field $\mathbb{C}$. A is said to be a *semi-topological algebra* if A is an algebra with a Hausdorff topology and if the maps: $(x, y) \longmapsto x + y$ and $(\lambda, x) \longmapsto \lambda x$ from $A \times A$ to A and $\mathbb{C}\times A$ to A, respectively, are continuous and the map: $(x, y) \longmapsto xy$ is separately continuous. A semi - topological algebra is said to be a *topological algebra* if the map: $(x, y) \longmapsto xy$ is jointly continuous. That is, an algebra with a topology is a *topological algebra* if it is a topological vector space in which multiplication is jointly continuous. Each topological vector space contains a base $\{U\}$ of zero neighborhoods such that each $U \in \{U\}$ is closed, circled, absorbing and for each $U \in \{U\}$ there is $V \in \{U\}$ such that $V + V \subset U$. If A is a topological algebra, then there is a base $\{U\}$ of zero neighborhoods satisfying these conditions and an additional condition: for each $U \in \{U\}$ there there are $V, W \in \{U\}$ such that $VW \subset U$. A B_o-algebra is a complete, metrizable locally convex algebra. Sometimes a locally convex algebra is described as a topological algebra whose topology is given by a family $\{p_\alpha\}_{\alpha\in\Gamma}$ of seminorms satisfying:

(i) $p_\alpha(\lambda x) = |\lambda|\, p_\alpha(x)$ for all $x \in A, \lambda \in \mathbb{C}$

(ii) $p_\alpha(x+y) \le p_\alpha(x) + p_\alpha(y)$ for all $x, y \in A$

(iii) $p_\alpha(x) = 0$ for all $\alpha \in \Gamma$ if and only if $x = 0$

(iv) for each $p_\alpha \in \{p_\alpha\}_{\alpha\in\Gamma}$ there is $p_\beta \in \{p_\alpha\}_{\alpha\in\Gamma}$ such that $p_\beta(xy) \le p_\alpha(x)p_\alpha(y)$ for all $x, y \in A$.

A locally convex algebra $(A, \{p_\alpha\}_{\alpha\in\Gamma})$ is said to be *multiplicatively convex* (or *locally $m-$convex* for short) if each $p_\alpha \in \{p_\alpha\}_{\alpha\in\Gamma}$ satisfies:

$$p_\alpha(xy) \le p_\alpha(x)p_\alpha(y) \quad \text{for all} \quad x, y \in A \quad \text{and} \quad \alpha \in \Gamma.$$

Every normed algebra is a locally $m-$convex algebra. A locally $m-$convex B_o-algebra is called a *Fréchet algebra.*

Let A be a locally convex algebra and let $L(A)$ be the vector space of all continuous linear maps on A. A map $T \in L(A)$ is said to be *Banach precompact* if and only if TB is precompact in A for every bounded subset B of A.

We are now ready to define the notion of a Banach precompact element in a locally convex algebra A. Let y be a fixed element of a locally convex algebra A. Then y is said to be *left Banach precompact* (*resp. right Banach precompact*) if the map $T_y := x \longmapsto yx$ (*resp.* $T_{,y} := x \longmapsto xy$) is Banach precompact on A. y is said to be (just) *Banach precompact* if the map $T_{y,y} := x \longmapsto yxy$ is Banach precompact on A. If every element $y \in A$ is (right) Banach precompact, then A is said to be a (right) *Banach precompact locally convex algebra.*

Theorem. *Let y be a Banach precompact element of a locally $m-$convex B_o- algebra A, then $A(y)$ is a Banach precompact locally convex algebra. ($A(y)$ is the least closed subalgebra of A containing y, which is the closure of the set of all polynomials in y without a constant term).*

Proof. Let $x \in A(y)$. Since A is a *Fréchet* algebra, there exists a sequence $\{x_n\}$ of elements which are polynomials in y, such that $\lim_n x_n = x$. Let B be a bounded subset of A. Define the operators T and T_n $(n = 1, 2, 3, \ldots)$ by

$$TB = xBx \quad \text{and} \quad T_nB = x_nBx_n$$

respectively.

Let $\{p_\alpha\}_{\alpha\in\Gamma}$ be a family of continuous seminorms generating the topology of A. For each $p_\alpha \in \{p_\alpha\}_{\alpha\in\Gamma}$ we have

$$\begin{aligned} p_\alpha(T_nB - TB) &= p_\alpha(x_nBx_n - xBx) \\ &= p_\alpha(x_nBx_n - x_nBx + x_nBx - xBx) \\ &= p_\alpha[x_nB(x_n - x) + (x_n - x)Bx] \\ &= p_\alpha[(x_n - x)(x_n + x)B] \\ &\le p_\alpha(x_n - x)[p_\alpha(x_n) + p_\alpha(x)]p_\alpha(B). \end{aligned}$$

Let $b \in B$, then there exists $\lambda > 0$ such that $p_\alpha(b) \le \lambda$. Since $\{x_n\}$ is bounded, there exists $\mu > 0$ such that $p_\alpha(x_n) \le \mu$ for all $n \in \mathbb{N}$. Therefore,

$$p_\alpha(T_nB - TB) \le \lambda p_\alpha(x_n - x)[\mu + p_\alpha(x)].$$

Hence,

$$\begin{aligned} \lim_n P_{B,\alpha}(T_n - T) &= \lim_n \sup_{b\in B} p_\alpha(T_nb - Tb) \\ &= 0. \end{aligned}$$

This implies that $T_n \longrightarrow T$ in the topology of bounded convergence on $L(A)$. Since the space of all Banach precompact operators on A is closed in $L(A)$ and since the operators $\{T_n : n \in \mathbb{N}\}$ are Banach precompact, it follows that T is Banach precompact as well. Therefore $A(y)$ is a Banach precompact algebra.

As a corollary we next show that a Banach algebra which is singly generated by a left Banach precompact element is a left Banach precompact algebra.

Corollary. *Let A be a Banach algebra and let $y \in A$ be a left Banach precompact element. If A is singly generated by y, then A is a left Banach precompact algebra.*

Proof. Let A be a Banach algebra and let y be a left Banach precompact element of A which generates A. Then $A = A(y)$, where $A(y)$ is the least closed subalgebra of A

containing y. The map $T_y := x \longmapsto yx$ is Banach precompact on A. Since $A(y)$ is the closure of the algebra of all polynomials of the form

$$p(y) = \lambda_1 y + \lambda_2 y^2 + \lambda_3 y^3 + \ldots + \lambda_n y^n,$$

where $n \in \mathbb{N}$ and $\lambda_i \geq 0$ $(i = 1, 2, 3, ..., n)$. Then, an element x of $A(y)$ is the limit of some polynomials $p_m(y)$ in y as $m \longrightarrow \infty$. Each $p_m(y)$ is a left Banach precompact element of A since the sum of Banach precompact elements is Banach precompact and the product of Banach precompact elements is Banach precompact. Therefore x is also a left Banach precompact element of A. This shows that A is a left Banach precompact algebra.

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