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ON GENERALIZED EXTREME–VALUE ORDER STATISTICS AND MOMENTS

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Abstract. Recurrence relations for, fractional single and product, moments of order statistics of a random variable drawn from the generalized extreme value distribution are obtained. The relations for fractional moments lead to some relations between negative moments of order statistics

1. INTRODUCTION

The Generalized Extreme Value (GEV) distribution combines into a single form the three possible types of limiting distribution of extremes, as derived by Fisher and Tippett [10]. Let the random variable X has distribution function

$$F(x) = \begin{cases} \exp[-\{1 - \dfrac{k(x-\epsilon)}{\sigma}\}^{\frac{1}{k}}], & k \neq 0 \\ \exp[-\exp\{-\dfrac{x-\epsilon}{\sigma}\}], & k = 0 \end{cases} \tag{1}$$

taken as the limit $k \to 0$ and the density function

$$f(x) = \begin{cases} \dfrac{1}{\sigma}\{1 - \dfrac{k(x-\epsilon)}{\sigma}\}^{\frac{1}{k}-1} exp[-\{1 - \dfrac{k(x-\epsilon)}{\sigma}\}^{\frac{1}{k}}], & k \neq 0 \\ \dfrac{1}{\sigma}[1 - \dfrac{(x-\epsilon)}{\sigma}] exp[-exp[-\dfrac{(x-\epsilon)}{\sigma}]], & k = 0 \end{cases} \tag{2}$$

where x is bounded by $\epsilon + \frac{\sigma}{k}$ from above if $k > 0$ and from below if $k < 0$. The parameters of the distribution are ϵ, the location parameter, σ, the scale paprameter, and k, the shape parameter. The latter is most important as it determines which extreme value distribution is represented: Fisher-Tippett Types I, II and III correspond to $k = 0, k < 0$ and $k > 0$, respectively. In practice, the shape parameter lies in the range $-\frac{1}{2} < k < \frac{1}{2}$, Hoskin et al. [11]. Balakrishnan et al [4] have discussed recurrence relations for moments of record values from the distribution.

In this paper, we look at the case $k \neq 0$ since the case $k = 0$ has infinite support and we seek recurrence relations for moments of order statistics from the distribution. It can be easily seen from (1) and (2) that

$$(1 - kx)^{1-\frac{1}{k}} f(x) = F(x) \tag{3}$$

Let $X_{1:n} \leq X_{2:n} \leq ..... \leq X_{n:n}$ be order statistics from the GEV distribution.

Let us denote

$$\mu_{r:n}^{(i)} = E(X_{r:n}^i), 1 \leq r \leq n \tag{4}$$

and

$$\mu_{r,s:n}^{(i,j)} = E(X_{r:n}^i X_{s:n}^j), 1 \leq r < s \leq n. \tag{5}$$

Also

$$f_{r:n}(x) = C_{r:n}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), \quad 1 \leq r \leq n, -\infty < x < \infty \tag{6}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ and

$$\begin{aligned} &f_{r,s:n}(x, y) = \\ &\quad C_{r,s:n}[F(x)]^{r-1}[f(x)][F(y) - F(x)]^{s-r-1} f(y)[1 - F(y)]^{n-s}.1 \leq r < s \leq n \end{aligned} \tag{7}$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$

It is of interest to point out that similar results are availavble for many other distributions for example Adeyemi [1, 2], Ali and Khan [3], Balakrishnan et al [5, 6, 7], Joshi [12, 13]e.t.c.

2. RECURRENCE RELATION FOR SINGLE MOMENTS

Theorem 2.1. *For* $-\frac{1}{2} < k < 0, \quad n \geq 2$ *and* $i = 1, 2, ...$

$$\mu_{2:n}^{(i(1-\frac{1}{k})+1)} = \frac{i(k-1)+k}{k(n-1)} \sum_{t=0}^{1-\frac{1}{k}} \binom{1-\frac{1}{k}}{t} (-k)^{1-\frac{1}{k}-t} [\mu_{1:n}^{(i)}]^{1-\frac{1}{k}} [\mu_{1:n}^{(i+1)}]^{1-\frac{1}{k}-t} + \frac{1}{n-1} \mu_{1:n}^{(i(1-\frac{1}{k})+1)} \tag{8}$$

and for $0 < k < \frac{1}{2}$

$$\mu_{2:n}^{(i(1-\frac{1}{k})+1)} = \frac{i(k-1)+k}{k(n-1)} (\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)})^{1-\frac{1}{k}} + \frac{1}{n-1} \mu_{1:n}^{(i(1-\frac{1}{k})+1)} \tag{9}$$

Proof.

$$\begin{aligned} (\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)})^{1-\frac{1}{k}} &= n \int_x x^{i(1-\frac{1}{k})} (1-kx)^{1-\frac{1}{k}} [1-F(x)]^{n-1} f(x) dx \\ &= n \int_x x^{i(1-\frac{1}{k})} F(x) [1-F(x)] dx \end{aligned} \tag{10}$$

having used (3), (4) and (6). Integrating (10) by parts, we obtain

$$(\mu_{1:n}^{(i)} - k\mu_{1:n}^{(i+1)})^{1-\frac{1}{k}} = \frac{k(n-1)}{i(k-1)+k} \mu_{2:n}^{(i(1-\frac{1}{k})+1)} - \frac{k}{i(k-1)+k} \mu_{1:n}^{(i(1-\frac{1}{k})+1)}. \tag{11}$$

The relations (8) and (9) are obtained by simply rewriting (11).

Corollary 2.1. *When* $k = 1, \frac{1}{2}$ *and* $-\frac{1}{2}$ *and for* $n \geq 2$ *we have, respectively*

$$(n-1)\mu_{2:n} = 1 + \mu_{1:n} \tag{12}$$

$$\mu_{2:n}^{(1-i)} = \frac{1-i}{n-1} [\mu_{1:n}^{(i)} - \frac{1}{2} \mu_{1:n}^{(i+1)}]^{-1} + \frac{1}{n-1} \mu_{1:n}^{(1-i)} \tag{13}$$

$$\mu_{2:n}^{(3i+1)} = \frac{3i+1}{8(n-1)} \sum_{t=0}^{3} \binom{3}{t} 2^t [\mu_{1:n}^{(i)}]^3 [\mu_{1:n}^{(i+1)}]^{3-t} + \frac{1}{n-1} \mu_{1:n}^{(3i+1)} \tag{14}$$

Theorem 2.2. *For* $-\frac{1}{2} < k < 0, \quad 1 \leq r \leq n-1$ *and* $i = 0, 1, 2, ...$

$$\mu_{r+1:n}^{(i(1-\frac{1}{k})+1)} = \frac{i(k-1)+k}{kr} \sum_{t=0}^{1-\frac{1}{k}} \binom{1-\frac{1}{k}}{t} (-k)^{1-\frac{1}{k}-t} [\mu_{r:n}^{(i)}]^{1-\frac{1}{k}} [\mu_{r:n}^{(i+1)}]^{1-\frac{1}{k}-t} \mu_{r:n}^{(i(1-\frac{1}{k})+1)} \tag{15}$$

and for $0 < k < \frac{1}{2}$

$$\mu_{r+1:n}^{(i(1-\frac{1}{k})+1)} = \frac{i(k-1)+k}{kr}(\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)})^{1-\frac{1}{k}} + \mu_{r:n}^{(i(1-\frac{1}{k})+1)} \tag{16}$$

Proof.

$$\begin{aligned}(\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)})^{1-\frac{1}{k}} \\ &= C_{r:n}\int_x x^{i(1-\frac{1}{k})}(1-kx)^{1-\frac{1}{k}}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \\ &= C_{r:n}\int_x x^{i(1-\frac{1}{k})}[F(x)]^{r}[1-F(x)]^{n-r}dx\end{aligned} \tag{17}$$

having used (3), (4) and (6) Integrating (16) by parts, we then have

$$(\mu_{r:n}^{(i)} - k\mu_{r:n}^{(i+1)})^{1-\frac{1}{k}} = \frac{kr}{i(k-1)+k}\mu_{r+1:n}^{(i(1-\frac{1}{k})+1)} - \frac{kr}{i(k-1)+k}\mu_{r:n}^{(i(1-\frac{1}{k})+1)} \tag{18}$$

The relations (15) and (16) are obtained by rewriting (18).

Corollary 2.2. *By setting* $k = 1, \frac{1}{2}$ *and* $-\frac{1}{2}$ *the results in* (15) *and* (16) *reduce to*

$$\mu_{r+1:n} = \frac{1}{r} + \mu_{r:n} \tag{19}$$

$$\mu_{+1:n}^{(3i+1)} = \frac{3i+1}{8r}\sum_{t=0}^{3}\binom{3}{t}2^t[\mu_{r:n}^{(i)}]^3[\mu_{r:n}^{(i+1)}]^{3-t} + \mu_{r:n}^{(3i+1)} \tag{20}$$

$$\mu_{r+1:n}^{(i-1)} = \frac{1-i}{r}(\mu_{r:n}^{(i)} - \frac{1}{2}\mu_{r:n}^{(i+1)})^{-1} + \mu_{r:n}^{(1-i)} \tag{21}$$

Theorem 2.3. *For* $-\frac{1}{2} < k < 0,\ r+k \leq n-1$ *and* $i = 1, 2, ...$

$$\begin{aligned}\mu_{r+k+1:n}^{(i(1-\frac{1}{k})+1)} = \frac{i(k-1)+k}{k(r+k)}\sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}[\mu_{r+k:n}^{(i)}]^{1-\frac{1}{k}}[\mu_{r+k:n}^{(i+1)}]^{1-\frac{1}{k}-t} \\ +\mu_{r+k:n}^{(i(1-\frac{1}{k})+1)}\end{aligned} \tag{22}$$

and for $0 < k < \frac{1}{2}$

$$\mu_{r+k+1:n}^{(i(1-\frac{1}{k})+1)} = \frac{i(k-1)+k}{k(r+k)}[\mu_{r+k:n}^{(i)} - k\mu_{r+k:n}^{(i+1)}]^{1-\frac{1}{k}} + \mu_{r+k:n}^{(i(1-\frac{1}{k})+1)} \tag{23}$$

Proof.

$$
\begin{aligned}
&[\mu_{r+k:n}^{(i)} - k\mu_{r+k:n}^{(i+1)}]^{1-\frac{1}{k}} \\
&\qquad = C_{r+k:n}\int_x (x^i - kx^{i+1})^{1-\frac{1}{k}}[F(x)]^{r+k-1}[1-F(x)]^{n-r-k}f(x)dx \\
&\qquad = C_{r+k:n}\int_x x^{i(1-\frac{1}{k})}[F(x)]^{r+k}[1-F(x)]^{n-r-k}dx
\end{aligned}
\tag{24}
$$

having used (3), (4) and (6).

Integrating (24) by parts and after simplification, we have

$$
[\mu_{r+k:n}^{(i)} - k\mu_{r+k:n}^{(i+1)}]^{1-\frac{1}{k}} = \frac{k(r+k)}{i(k-1)} + k\mu_{r+k+1:n}^{(i(1-\frac{1}{k})+1)} - \frac{k(r+k)}{i(k-1)+k}\mu_{r+k:n}^{(i(1-\frac{1}{k})+1)}. \tag{25}
$$

By rewriting (25), the relations (22) and (23) are obtained.

Corollary 2.3. *By setting $k = 1, \frac{1}{2}$ and $-\frac{1}{2}$ the results in (22) and (23) respectively yield*

$$
\mu_{r+2:n} = \frac{1}{r} + \mu_{r+1:n} \tag{26}
$$

$$
\mu_{r+k+1:n}^{(1-i)} = \frac{1-i}{r+k}[\mu_{r+k:n}^{i} - \frac{1}{2}\mu_{r+k:n}^{(i+1)}]^{-1} + \mu_{r+k:n}^{(1-i)}\ ,\ i \geq 2 \tag{27}
$$

$$
\mu_{r+k+1:n}^{(3i+1)} = \frac{3i+1}{8(r+k)}\sum_{t=0}^{3}\binom{3}{t}2^t[\mu_{r+k:n}^{(i)}]^3[\mu_{r+k:n}^{(i+1)}]^{3-t} + \mu_{r+k:n}^{(3i+1)} \tag{28}
$$

Theorem 2.4 *For $-\frac{1}{2} < k < 0$, $r+k \leq n-1$ and $i = 1, 2, ...$*

$$
\begin{aligned}
&\mu_{r+1:n}^{(i(1-\frac{1}{k})-k+2)} \\
&\quad = \frac{i(k-1)-k^2+2k}{kr}\sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}[\mu_{r:n}^{(i-k)}]^{1-\frac{1}{k}}[\mu_{r:n}^{(i-k+1)}]^{1-\frac{1}{k}-t} \\
&\qquad + \mu_{r:n}^{(i(1-\frac{1}{k})-k+2)}
\end{aligned}
\tag{29}
$$

and for $0 < k < \frac{1}{2}$

$$
\mu_{r+1:n}^{(i(1-\frac{1}{k})-k+2)} = \frac{i(k-1)-k^2+2k}{kr}[\mu_{r:n}^{(i-k)} - k\mu_{r:n}^{(i-k+1)}]^{1-\frac{1}{k}} + \mu_{r:n}^{(i(k-1)-k+2)} \tag{30}
$$

Proof

$$\begin{aligned}[\mu_{r:n}^{(i-k)} - k\mu_{r:n}^{(i-k+1)}]^{1-\frac{1}{k}} \\ &= C_{r:n}\int_x x^{i(1-\frac{1}{k})-k+2}(1-kx)^{1-\frac{1}{k}}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx \\ &= C_{r:n}\int_x x^{i(1-\frac{1}{k})-k+2}[F(x)]^{r}[1-F(x)]^{n-r}dx\end{aligned} \tag{31}$$

having used (3), (4) and (6).

Integrating (31) by parts and after simplification, we have

$$\begin{aligned}&[\mu_{r:n}^{(i-k)} - k\mu_{r:n}^{(i-k+1)}]^{1-\frac{1}{k}} \\ &= \frac{kr}{i(k-1)-k^2+2k}\mu_{r+1:n}^{(i(1-\frac{1}{k})-k+2)} - \frac{kr}{i(1-\frac{1}{k})-k^2+2k}\mu_{r:n}^{(i(1-\frac{1}{k})-k+2)}.\end{aligned} \tag{32}$$

By simply rewriting (32), we have the relations (29) and (30).

Corollary 2.4. *By setting* $k=\frac{1}{2}$ *and* $-\frac{1}{2}$ *the results in* (29) *and* (30) *respectively reduce to*

$$\mu_{r+1:n}^{(3i+\frac{9}{2})} = \frac{6i+9}{16r}\sum_{t=0}^{3}\binom{3}{t}2^t[\mu_{r:n}^{(i+\frac{1}{2})}]^3[\mu_{r:n}^{(i+\frac{3}{2})}]^{3-t} + \mu_{r:n}^{(3i+\frac{9}{2})} \tag{33}$$

and

$$\mu_{r+1:n}^{(\frac{3}{2}-i)} = \frac{(\frac{3}{2}-i)}{r}[\mu_{r:n}^{(i-\frac{1}{2})} - \frac{1}{2}\mu_{r:n}^{i+\frac{1}{2}}]^{-1} + \mu_{r:n}^{(\frac{3}{2}-i)} \tag{34}$$

3. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

Theorem 3.1. *For* $-\frac{1}{2}<k<0$, *and* $1\le r\le n-2$

$$\begin{aligned}\mu_{r,r+2:n}^{(1-\frac{1}{k},1)} = \sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}(\mu_{r:n})^{1-\frac{1}{k}}(\mu_{r,r+1:n})^{1-\frac{1}{k}-t} \\ -r\mu_{r+1}^{(1-\frac{1}{k}+1)} - \mu_{r,r+1:n-1}^{(1-\frac{1}{k},1)} + \mu_{r,r+1:n}^{(1-\frac{1}{k},1)},\end{aligned} \tag{35}$$

and for $0<k<\frac{1}{2}$

$$\mu_{r,r+2:n}^{(1-\frac{1}{k},1)} = [\mu_{r:n} - k\mu_{r,r+1:n}]^{1-\frac{1}{k}} - r\mu_{r+1}^{(1-\frac{1}{k}+1)} - \mu_{r,r+1:n-1}^{(1-\frac{1}{k},1)} + \mu_{r,r+1:n}^{(1-\frac{1}{k},1)} \tag{36}$$

Proof

$$\begin{aligned}[\mu_{r:n}-k\mu_{r,r+1:n}]^{1-\frac{1}{k}} &= C_{r,r+1:n}\int\int_{x<y}(x-kxy)^{1-\frac{1}{k}}[F(x)]^{r-1}[1-F(y)]^{n-r-1} \\ &\qquad \times f(x)f(y)dxdy \qquad (37)\\ &= C_{r,r+1:n}\int_x x^{1-\frac{1}{k}}[F(x)]^{r-1}f(x)I_1(x)dx\end{aligned}$$

having used (3), (5) and (7) where

$$I_1(x)=\int_y [F(y)][1-F(y)]^{n-r-1}dy.$$

Integrating $I_1(x)$ by parts and substituting in (37), we have

$$\begin{aligned}&[\mu_{r:n}-k\mu_{r,r+1:n}]^{1-\frac{1}{k}}\\ &\quad = C_{r,r+1:n}\int_x x^{2-\frac{1}{k}}[F(x)]^{r}[1-F(x)]^{n-r-1}f(x)dx\\ &\quad +C_{r,r+1:n}\int\int_{x<y}x^{1-\frac{1}{k}}y[F(x)]^{r-1}[F(y)-F(x)][1-F(y)]^{n-r-2}f(x)f(y)dxdy\\ &\quad +C_{r,r+1:n}\int\int_{x<y}x^{1-\frac{1}{k}}y[F(x)]^{r-1}[1-F(y)]^{n-r-2}f(x)f(y)dxdy\\ &\quad -C_{r,r+1:n}\int\int_{x<y}x^{1-\frac{1}{k}}y[F(x)]^{r-1}[1-F(y)]^{n-r-1}f(x)f(y)dxdy.\end{aligned}$$

By simplifying the above expressions, we obtain our results in (35) and (36).

Corollary 3.1. *Setting* $k=-\frac{1}{2}$ *and* 1*, we obtain*

$$\mu^{(3)}_{r,r+2:n}=\frac{1}{8}\sum_{t=0}^{3}\binom{3}{t}2^t(\mu_{r:n})^3(\mu_{r,r+1:n})^{3-t}-r\mu^{(4)}_{r+1:n}-\mu^{(3)}_{r,r+1:n-1}-\mu^{(3)}_{r,r+1:n} \qquad (38)$$

and

$$\mu_{r+2:n}=(1-r)\mu_{r+1:n}-\mu_{r+1:n-1}. \qquad (39)$$

Theorem 3.2. *For* $-\frac{1}{2}<k<0$, *and* $1\le r\le n-1$

$$\mu^{(1,1-\frac{1}{k})}_{r,r+1:n}=r\mu^{(1-\frac{1}{k}+1)}_{r+1:n}-\sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}(\mu_{r+1:n})^{1-\frac{1}{k}}(\mu_{r,r+1:n})^{1-\frac{1}{k}-t} \qquad (40)$$

and for $0<k<\frac{1}{2}$

$$\mu^{(1,1-\frac{1}{k})}_{r,r+1:n}=r\mu^{(1-\frac{1}{k}+1)}_{r+1:n}-[\mu_{r+1:n}-k\mu_{r,r+1:n}]^{1-\frac{1}{k}} \qquad (41)$$

Proof

$$\begin{aligned}
&[\mu_{r+1:n} - k\mu_{r,r+1:n}]^{1-\frac{1}{k}} \\
&\quad = C_{r,r+1:n}\int\int_{x<y}(y-kxy)^{1-\frac{1}{k}}[F(x)]^{r-1}[1-F(y)]^{n-r-1}f(x)f(y)dxdy \\
&\quad = C_{r,r+1:n}\int_y y^{1-\frac{1}{k}}[1-F(y)]^{n-r-1}f(y)I_2(y)dy
\end{aligned} \tag{42}$$

having used $(3),(5)$ and (7) where

$$I_2(y) = \int_x [F(x)]^r dx.$$

Integrating $I_2(y)$ by parts, we have

$$I_2(y) = y[F(y)]^r - r\int_x x[F(x)]^{r-1}f(x)dx.$$

Upon substituting in (42) and simplifying the resulting expression we obtain the relations (40) and (41).

Corollary 3.2. *Setting $k = -\frac{1}{2}$ and $\frac{1}{2}$ we have*

$$\mu_{r,r+1:n}^{(1,3)} = r\mu_{r+1:n}^{(4)} - \frac{1}{8}\sum_{t=0}^{3}\binom{3}{t}2^t(\mu_{r+1:n})^3(\mu_{r,r+1:n})^{3-t} \tag{43}$$

and

$$\mu_{r,r+1:n}^{(1,-1)} = -\frac{1}{\mu_{r+1:n} - \frac{1}{2}\mu_{r,r+1:n}} \tag{44}$$

Remark: The expression (44) is a relationship between negative and positive moments.

Theorem 3.3. *For $-\frac{1}{2} < k < 0$, and $1 \leq r < s \leq n-1$*

$$\begin{aligned}
\mu_{r+1,s+1:n}^{(1-\frac{1}{k},1)} = \frac{1}{r}\sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}(\mu_{r:n})^{1-\frac{1}{k}}(\mu_{r,s:n})^{1-\frac{1}{k}-t} \\
-\frac{s-r}{r}[\mu_{r,s+1:n}^{(1-\frac{1}{k},1)} - \mu_{r,s:n}^{1-\frac{1}{k},1}] + \frac{s-r-1}{r}\mu_{r+1,s:n}
\end{aligned} \tag{45}$$

and for $0 < k < \frac{1}{2}$

$$\mu^{(1-\frac{1}{k},1)}_{r+1,s+1:n} = \frac{[\mu_{r:n} - k\mu_{r,s:n}]^{1-\frac{1}{k}}}{r} - \frac{s-r}{r}[\mu^{(1-\frac{1}{k},1)}_{r,s+1:n} - \mu^{1-\frac{1}{k},1}_{r,s:n}] + \frac{s-r-1}{r}\mu_{r+1,s:n}. \quad (46)$$

Proof.

$$\begin{aligned}[\mu_{r:n} - k\mu_{r,s:n}]^{1-\frac{1}{k}} &= C_{r,s:n}\int\int_{x<y}(x-kxy)^{1-\frac{1}{k}}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} \\ &\quad \times[1-F(y)]^{n-s}f(x)f(y)dxdy \\ &= C_{r,s:n}\int_x x^{1-\frac{1}{k}}[F(x)]^{r-1}I_3(x)f(x)dx\end{aligned} \quad (47)$$

having used (3), (5) and (7), where

$$I_3(x) = \int_y [F(y)][F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dx.$$

Integrating $I_3(x)$ by parts, we have

$$\begin{aligned}I_3(x) &= (n-s)\int_y y[F(y)][F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1}f(y)dy \\ &\quad -(s-r-1)\int_y y[F(y)][F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(y)dy \\ &\quad -\int_y y[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy.\end{aligned} \quad (48)$$

By substituting (48) into (47) and simplifying, we have the relation (45) and (46).

Corollary 3.3. *Setting* $k = -\frac{1}{2}$ *and* 1 *we have*

$$\begin{aligned}\mu^{(3,1)}_{r+1,s+1:n} &= \frac{1}{8r}\sum_{t=0}^{3}\binom{3}{t}2^t(\mu_{r:n})^3(\mu_{r,s:n})^{3-t} - \frac{s-r}{r}[\mu^{(3,1)}_{r,s+1:1} - \mu^{(3,1)}_{r,s:n}] \\ &\quad + \frac{s-r-1}{r}\mu^{(3,1)}_{r+1,s:n}\end{aligned} \quad (49)$$

and

$$\mu_{s+1:n} = \frac{1}{s} + \frac{2(s-r)-1}{s}\mu_{s:n}. \quad (50)$$

Theorem 3.4. *For* $-\frac{1}{2} < k < 0$, *and* $1 \le r < s \le n$

$$\mu^{(1,1-\frac{1}{k})}_{r+1,s:n} = \frac{1}{r}\sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}(\mu_{s:n})^{1-\frac{1}{k}}(\mu_{r,s:n})^{1-\frac{1}{k}-t} + \mu^{(1,1-\frac{1}{k})}_{r,s:n} \quad (51)$$

and for $0 < k < \frac{1}{2}$

$$\mu_{r+1,s:n}^{(1,1-\frac{1}{k})} = \frac{[\mu_{s:n} - k\mu_{r,s:n}]^{1-\frac{1}{k}}}{r} + \mu_{r,s:n}^{(1,1-\frac{1}{k})}. \tag{52}$$

Proof

$$\begin{aligned}[\mu_{s:n} - k\mu_{r,s:n}]^{1-\frac{1}{k}} &= C_{r,s:n}\int\int_{x<y}(y-kxy)^{1-\frac{1}{k}}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1} \\ &\qquad \times[1-F(y)]^{n-s}f(x)f(y)dxdy \\ &= C_{r,s:n}\int_y y^{1-\frac{1}{k}}[1-F(y)]^{n-s}I_4(y)f(y)dy\end{aligned} \tag{53}$$

having used (3), (5) and (7) where

$$I_4(y) = \int_x [F(x)]^r[F(y)-F(x)]^{s-r-1}dx.$$

Integrating $I_4(y)$ by parts, we have

$$\begin{aligned}I_4(y) = (s-r-1)&\int_x x[F(x)]^r[F(y)-F(x)]^{s-r-2}f(x)dx \\ -r&\int_x x[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}f(x)dx.\end{aligned} \tag{54}$$

Upon substituting (54) into (53), and after simplification we obtain the relations (51) and (52).

Corollary 3.4. *By setting* $k = -\frac{1}{2}$ *we have*

$$\mu_{r+1,s:n}^{(1,3)} = \frac{1}{8r}\sum_{t=0}^{3}\binom{3}{t}2^t(\mu_{s:n})^3(\mu_{r,s:n})^{3-t} + \mu_{r,s:n}^{(1,1-\frac{1}{k})}. \tag{55}$$

Theorem 3.5. *For* $-\frac{1}{2} < k < 0$, *and* $1 \le r < s \le n-2$

$$\begin{aligned}\mu_{r+1,r+2:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} = \sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}(\mu_{r,r+1:n}^{(i,i)})^{1-\frac{1}{k}}(\mu_{r,r+1}^{i,i+1})^{1-\frac{1}{k}-t} - \mu_{r+1:n}^{(2i(-\frac{1}{k})+1)} \\ -\frac{1}{r}\mu_{r,r+2:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} + \frac{1}{r}\mu_{r,r+1:n}^{(i-\frac{i}{k},i-\frac{i}{k})}\end{aligned} \tag{56}$$

and for $0 < k < \frac{1}{2}$

$$\begin{aligned}\mu_{r+1,r+2:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} = \frac{(1-\frac{1}{k})i+1}{r}[\mu_{r,r+1:n}^{(i,i)} - k\mu_{r,r+1:n}^{(i,i+1)}]^{1-\frac{1}{k}} - \mu_{r+1:n}^{2i(1-\frac{1}{k})+1)} \\ -\frac{1}{r}\mu_{r,r+2:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} + \frac{1}{r}\mu_{r,r+1:n}^{(i-\frac{i}{k},i-\frac{i}{k})}\end{aligned} \tag{57}$$

Proof.

$$\begin{aligned}[\mu_{r,r+1:n}^{(i,i)} - k\mu_{r,r+1:n}^{(i,i+1)}]^{1-\frac{1}{k}} &= C_{r,r+1:n}\int\int_{x<y} x^{i(1-\frac{1}{k})} y^{i(1-\frac{1}{k})}(1-ky)^{1-\frac{1}{k}}[F(x)]^{r-1} \\ &\quad \times[1-F(y)]^{n-r-1} f(x)f(y)dxdy \\ &= C_{r,r+1:n}\int_x x^{i(1-\frac{1}{k})}[F(x)]^{r-1} I_5(x) f(x)dx \end{aligned} \tag{58}$$

having used $(3), (5)$ and (7) where

$$I_5(x) = \int_y y^{i(1-\frac{1}{k})}[F(y)][1-F(y)]^{n-r-1}dy,$$

which upon integrating by parts leads to

$$\begin{aligned} I_5(x) &= \frac{1}{i(1-\frac{1}{k})+1}\int_y x^{i(1-\frac{1}{k})+1}[F(x)][1-F(x)]^{n-r-1} \\ &\quad + \frac{n-r-1}{i(1-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[F(y)][1-F(y)]^{n-r-2} f(y)dy \\ &\quad - \frac{1}{i(1-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[1-F(y)]^{n-r-1} f(y)dy. \end{aligned}$$

Putting the above expression into (58) and simplifying the resulting expression, we have the relations (56) and (57).

Corollary 3.5. *By setting $k = -\frac{1}{2}$ we have*

$$\begin{aligned} \mu_{r+1,r+2:n}^{(3i,3i+1)} &= \frac{3i+1}{8r}\sum_{t=0}^{3}\binom{3}{t} 2^t(\mu_{r,r+1:n}^{(i,i)})^3(\mu_{r,r+1:n}^{(i,i+1)})^{3-t} - \mu_{r+1:n}^{(6i+1)} \\ &\quad - \frac{1}{r}\mu_{r,r+2:n}^{(3i,3i+1)} + \frac{1}{r}\mu_{r,r+1:n}^{(3i,3i)} \end{aligned} \tag{59}$$

Theorem 3.6. *For $-\frac{1}{2} < k < 0$, and $1 \le r < s \le n-1$*

$$\begin{aligned} \mu_{r+1,s+1:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} &= \frac{i(1-\frac{1}{k})+1}{r}\sum_{t=0}^{1-\frac{1}{k}}\binom{1-\frac{1}{k}}{t}(-k)^{1-\frac{1}{k}-t}[\mu_{r,s:n}^{(i,i)}]^{1-\frac{1}{k}}[\mu_{r,s:n}^{(i,i+1)}]^{1-\frac{1}{k}-t} \\ &\quad - \frac{s-r}{r}\mu_{r,s+1:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} + \frac{i(1-\frac{1}{k})+s-r}{r}\mu_{r,s:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} \\ &\quad + \mu_{r+1,s:n}^{(i-\frac{i}{k},i-\frac{i}{k}+1)} \end{aligned} \tag{60}$$

and for $0 < k < \frac{1}{2}$

$$\begin{aligned}\mu^{(i-\frac{i}{k},i-\frac{i}{k}+1)}_{r+1,s+1:n} &= \frac{i(1-\frac{1}{k})+1}{r}[\mu^{(i,i)}_{r,s:n} - k\mu^{(i,i+1)}_{r,s:n}]^{1-\frac{1}{k}} - \frac{s-r}{r}\mu^{(i-\frac{i}{k},i-\frac{i}{k}+1)}_{r,s+1:n} \\ &+ \frac{i(1-\frac{1}{k})+s-r}{r}\mu^{(i-\frac{i}{k},i-\frac{i}{k}+1)}_{r,s:n} + \mu^{(i-\frac{i}{k},i-\frac{i}{k}+1)}_{r+1,s:n}.\end{aligned} \tag{61}$$

Proof

$$\begin{aligned}[\mu^{(i,i)}_{r,s:n} - k\mu^{(i,i+1)}_{r,s:n}]^{1-\frac{1}{k}} &= C_{r,s:n}\int\int_{x<y} x^{i(1-\frac{1}{k})}y^{i(1-\frac{1}{k})}(1-ky)^{1-\frac{1}{k}}[F(x)]^{r-1} \\ &\quad\times[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)dxdy \\ &= C_{r,s:n}\int_x x^{i(1-\frac{1}{k})}[F(x)]^{r-1}I_6(x)f(x)dx\end{aligned} \tag{62}$$

having used (3), (5) and (7) where

$$I_6(X) = \int_y y^{i(1-\frac{1}{k})}[F(y)][F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}dy,$$

which upon integrating by parts becomes

$$\begin{aligned}I_6(x) &= \frac{n-s}{i(1-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[F(y)-F(x)]^{s-r}[1-F(y)]^{n-s-1}f(y)dy \\ &+\frac{n-s}{i(1-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[F(x)][F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s-1}f(y)dy \\ &-\frac{1}{i(1-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy \\ &-\frac{s-r-1}{i(1-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy \\ &-\frac{s-r-1}{i(-\frac{1}{k})+1}\int_y y^{i(1-\frac{1}{k})+1}[F(x)][F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}f(y)dy.\end{aligned} \tag{63}$$

Upon subsituting the above expression in (62) and simplifying the resulting expression we have the relations (60) and (61).

Corollary 3.6. *By setting* $k = -\frac{1}{2}$*, we have*

$$\begin{aligned}\mu^{(3i,3i+1)}_{r+1,s+1:n} &= \frac{3i+1}{8r}\sum_{t=0}^{3}\binom{3}{t}2^t[\mu^{(i,i)}_{r,s:n}]^3[\mu^{(i,i+1)}_{r,s:n}]^{3-t} - \frac{s-r}{r}\mu^{(3i,3i+1)}_{r,s+1:n} \\ &+\frac{3i+s-r}{r}\mu^{(3i,3i+1)}_{r,s:n} + \mu^{(3i,3i+1)}_{r+1,s:n}.\end{aligned} \tag{64}$$

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