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# ON THE MIDPOINT QUADRATURE FORMULA FOR LIPSCHITZIAN MAPPINGS AND APPLICATIONS

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ABSTRACT. The estimation of the remainder term in midpoint formula for$L-$lipschitzian mappings is given. Applications for special means are also pointed out.

## 1. Introduction

The following inequality is well known in the literature as the *midpoint inequality*:

$$\left|\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a)\right| \leq \frac{1}{24} \parallel f'' \parallel_\infty (b-a)^3, \tag{1.1}$$

where the mapping $f : [a,b] \to R$ is supposed to be twice differentiable on the interval $(a,b)$ and having the second derivative bounded on $(a,b)$, that is

$$\parallel f'' \parallel_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty.$$

Now, if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and $f$ is as above, then we have the *midpoint quadrature formula*:

$$\int_a^b f(x)dx = A_M(f, I_h) + R_M(f, I_h) \tag{1.2}$$

where $A_M(f, I_h)$ is the *midpoint rule*

$$A_M(f, I_h) =: \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \tag{1.3}$$

and the *remainder term* $R_M(f, I_h)$ satisfies the estimation

$$|R_M(f,I_h)| \leq \frac{1}{24} \parallel f'' \parallel_\infty \sum_{i=0}^{n-1} h_i^3 \tag{1.4}$$

where $h_i := x_{i+1} - x_i$ for $i = 0, ..., n-1$.
When we have an equidistant partitioning of $[a,b]$ given by

$$I_n : x_i := a + \frac{b-a}{n} i, \ \ i = 0, ..., n, \tag{1.5}$$

then we have the formula

$$\int_a^b f(x)dx = A_{M,n}(f) + R_{M,n}(f) \tag{1.6}$$

where

$$A_{M,n}(f) := \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot \frac{2i+1}{2}\right) \tag{1.7}$$

and the remainder satisfies the estimation

$$|R_{M,n}(f)| \leq \frac{1}{24} \cdot \frac{(b-a)^3}{n^2} \parallel f'' \parallel_\infty . \tag{1.8}$$

For other midpoint type's inequalities see the recent book [1].

## 2. Midpoint inequality for lipschitzian mappings

The following midpoint inequality for lipschitzian mappings holds:

**THEOREM 2.1.** *Let $f : [a,b] \to R$ be an $L-$lipschitzian mapping on $[a,b]$. Then we have the inequality*

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{4} L (b-a)^2. \tag{2.1}$$

*The constant $\frac{1}{4}$ is the best possible one.*

*Proof.* Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^b p(x)df(x) = f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(x)dx \tag{2.2}$$

where

$$p(x) := \begin{cases} x-a \text{ if } x \in [a, \frac{a+b}{2}) \\ x-b \text{ if } x \in [\frac{a+b}{2}, b]. \end{cases}$$

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i \in \{0,...,n-1\}} (x_{i+1}^{(n)} - x_i^{(n)})$

and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$. If $p : [a,b] \to R$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to R$ is $L$-lipschitzian on $[a,b]$, then

$$\left| \int_a^b p(x) dv(x) \right| = \left| \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)})[v(x_{i+1}^{(n)}) - v(x_i^{(n)})] \right|$$

$$\leq \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})|(x_{i+1}^{(n)} - x_i^{(n)}) \left| \frac{v(x_{i+1}^{(n)}) - v(x_i^{(n)})}{x_{i+1}^{(n)} - x_i^{(n)}} \right|$$

$$\leq L \lim_{\nu(\Delta_n) \to 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})|(x_{i+1}^{(n)} - x_i^{(n)}) = L \int_a^b |p(x)| dx. \tag{2.3}$$

Applying the inequality (2.3) for $p(x)$ as above and $v(x) = f(x), x \in [a,b]$, we get

$$\left| \int_a^b p(x) df(x) \right| \leq L \int_a^b |p(x)| dx = L \frac{(b-a)^2}{4} \tag{2.4}$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).
Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left| \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq CL(b-a)^2. \tag{2.5}$$

Consider the mapping $f : [a,b] \to R,\ f(x) = |x - \frac{a+b}{2}|$. Then

$$|f(x) - f(y)| = ||x - \frac{a+b}{2}| - |y - \frac{a+b}{2}|| \leq |x - y|$$

for all $x, y \in [a,b]$, which shows that $f$ is $L-$lipschitzian with the constant $L = 1$.
For this mapping we have

$$\int_a^b f(x) dx - f\left(\frac{a+b}{2}\right)(b-a) = \frac{(b-a)^2}{4}$$

and

$$L(b-a)^2 = (b-a)^2$$

whence by (2.5) we get

$$\frac{(b-a)^2}{4} \leq C(b-a)^2.$$

This implies that $C \geq \frac{1}{4}$ and the sharpness of (2.1) is proved.
The following corollary holds:

**COROLLARY 2.2.** *Let $f : [a,b] \to R$ be a differentiable mapping on $(a,b)$ whose derivative is bounded on $(a,b)$. Then we have the inequality:*

$$\left| \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{4} \parallel f' \parallel_\infty (b-a)^2. \tag{2.6}$$

**Remark 2.3.** It is well known that if $f : [a, b] \to R$ is a convex mapping on $[a, b]$ , then *Hermite-Hadamard's* inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f : I \subset R \to R$ is convex on $I$ and $a, b \in Int(I),\ a < b$, then $f'_+$ is monotonous nondecreasing on $[a, b]$. By Theorem 2.1 we get

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}f'_+(b)(b-a) \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

The following corollary for midpoint composite formula holds:

**COROLLARY 2.4.** *Let $f : [a, b] \to R$ be an $L-$lipschitzian mapping on $[a, b]$ and $I_h$ a partition of $[a, b]$. Then we have the midpoint quadrature formula (1.2) and the remainder term $R_M(f, I_h)$ satisfies the estimation:*

$$|R_M(f, I_h)| \leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2. \tag{2.9}$$

*Moreover, the constant $\frac{1}{4}$ is the best possible one.*

*Proof.* Applying inequality (2.1) on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ we have

$$\left|\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right)h_i\right| \leq \frac{1}{4}Lh_i^2.$$

Using the generalized triangle inequality we get

$$|R_M(f, I_h)| = \left|\sum_{i=0}^{n-1}\left(\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right)h_i\right)\right|$$

$$\leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right)h_i\right| \leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2$$

and the corollary is proved.

The case of equidistant partitioning is embodied in the following corollary:

**COROLLARY 2.5.** *Let $I_n$ be an equidistant partitioning of $[a, b]$ and $f$ be as in Theorem 2.1. Then we have the formula (1.6) and the remainder satisfies the estimation:*

$$|R_{M,n}(f)| \leq \frac{1}{4} \cdot \frac{L}{n}(b-a)^2. \tag{2.10}$$

**Remark 2.6.** If we want to approximate the integral $\int_a^b f(x)dx$ by midpoint formula $A_{M,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division $I_n$, where

$$n_\varepsilon := \left[\frac{1}{4} \cdot \frac{L}{\varepsilon}(b-a)^2\right] + 1$$

and $[r]$ denotes the integer part of $r \in R$.

**Comments 2.7.** If the mapping $f : [a, b] \to R$ is neither twice differentiable nor the second derivative is bounded on $(a, b)$, then we can not apply the classical estimation in midpoint formula using the second derivative. But if we assume that $f$ is lipschitzian, then we can use instead the formula (2.9).

We give here a class of mappings which are lipschitzian but having the second derivative unbounded on the given interval.

Let $f_{p,q} : [a, b] \to R$, $f_{p,q}(x) := (x^q - a^q)^p$ where $p \in (1, 2)$ and $q \geq 2$. Then obviously

$$f'_{p,q}(x) := pqx^{q-1}(x^q - a^q)^{p-1},\ x \in (a, b)$$

and

$$f''_{p,q}(x) = pq\frac{x^{q-2}[(pq-1)x^q - (q-1)a^q]}{(x^q - a^q)^{2-p}},\ x \in (a, b).$$

It is clear that $f$ is lipschitzian with the constant

$$L = \| f'_{p,q} \|_\infty = pqb^{q-1}(b^q - a^q)^{p-1} < \infty$$

but $\lim_{x \to a+} f''_{p,q}(x) = +\infty$.

## 3. Applications for special means

Let us recall the following means:

1. *Arithmetic mean*

$$A = A(a, b) := \frac{a+b}{2},\ a, b \geq 0;$$

2. *Geometric mean*

$$G = G(a, b) := \sqrt{ab},\ a, b \geq 0;$$

3. *Harmonic mean*

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}},\ a, b > 0;$$

4. *Logarithmic mean*

$$L = L(a, b) := \frac{b-a}{\ln b - \ln a},\ a, b > 0,\ a \neq b;$$

5. *Identric mean*

$$I = I(a, b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}},\ a, b > 0,\ a \neq b;$$

6. *p-Logarithmic mean*

$$L_p = L_p(a,b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \ p \in R\backslash\{-1,0\}, \ a,b>0, \ a \neq b.$$

It is well known that $L_p$ is monotonous nondecreasing over $p \in R$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \le G \le L \le I \le A. \tag{3.1}$$

In what follows, by the use of Theorem 2.1, we point out some new inequalities for the above means.

**1.** Let $f : [a,b] \to R$ $(0 < a < b)$, $f(x) = x^p$, $p \in R\backslash\{-1,0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L_p(a,b), \ f\left(\frac{a+b}{2}\right) = A^p(a,b),$$

$$\|f'\|_\infty = \delta_p(a,b) := \begin{cases} pb^{p-1} & \text{if} \quad p \ge 1 \\ |p|a^{p-1} & \text{if} \quad p \in (-\infty,1)\backslash\{-1,0\}. \end{cases}$$

Using the inequality (2.6) we get

$$|L_p^p(a,b) - A^p(a,b)| \le \frac{1}{4}\delta_p(a,b)(b-a). \tag{3.2}$$

**2.** Let $f : [a,b] \to R$ $(0 < a < b)$, $f(x) = \dfrac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L^{-1}(a,b), \ f\left(\frac{a+b}{2}\right) = A^{-1}(a,b), \ \|f'\|_\infty = \frac{1}{a^2}.$$

Using the inequality (2.6) we get

$$0 \le A - L \le \frac{b-a}{4a^2}LA. \tag{3.3}$$

**3.** Let $f : [a,b] \to R$ $(0 < a < b)$, $f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = \ln I(a,b), \ f\left(\frac{a+b}{2}\right) = \ln A(a,b), \ \|f'\|_\infty = \frac{1}{a}.$$

Using the inequality (2.6) we get

$$1 \le \frac{A}{I} \le \exp\left(\frac{b-a}{4a}\right). \tag{3.4}$$

**References**

[1] Mitrinovć, D.S.; Pečarić, J.E. and Fink, A.M.: *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.