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ON THE MIDPOINT QUADRATURE FORMULA FOR MAPPINGS WITH BOUNDED VARIATION AND APPLICATIONS

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ABSTRACT. The estimation of the remainder term in midpoint formula for mappings with bounded variation is given. Applications for special means are also pointed out.

1. Introduction

The following inequality is well known in the literature as the *midpoint inequality*:

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \le \frac{1}{24} \parallel f'' \parallel_\infty (b-a)^3 \tag{1.1}$$

where the mapping $f : [a,b] \to R$ is supposed to be twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), that is

$$\parallel f'' \parallel_\infty := \sup_{x\in(a,b)} |f''(x)| < \infty.$$

Now, if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we have the *midpoint quadrature formula*:

$$\int_a^b f(x)dx = A_M(f, I_h) + R_M(f, I_h) \tag{1.2}$$

where $A_M(f, I_h)$ is the *midpoint rule*

$$A_M(f, I_h) =: \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \tag{1.3}$$

and the *remainder term* $R_M(f, I_h)$ satisfies the estimation

$$|R_M(f, I_h)| \leq \frac{1}{24} \parallel f'' \parallel_\infty \sum_{i=0}^{n-1} h_i^3 \tag{1.4}$$

where $h_i := x_{i+1} - x_i$ for $i = 0, ..., n-1$.
When we have an equidistant partitioning of $[a, b]$ given by

$$I_n : x_i := a + \frac{b-a}{n} i, \ \ i = 0, ..., n, \tag{1.5}$$

then we have the formula

$$\int_a^b f(x)dx = A_{M,n}(f) + R_{M,n}(f) \tag{1.6}$$

where

$$A_{M,n}(f) := \frac{b-a}{n} \sum_{i=0}^{n-1} f\left(a + \frac{b-a}{n} \cdot \frac{2i+1}{2}\right) \tag{1.7}$$

and the remainder satisfies the estimation

$$|R_{M,n}(f)| \leq \frac{1}{24} \cdot \frac{(b-a)^3}{n^2} \parallel f'' \parallel_\infty . \tag{1.8}$$

For other midpoint type's inequalities see the recent book [1].

2. Midpoint inequality for mappings with bounded variation

The following midpoint inequality for mappings with bounded variation holds:

THEOREM 2.1. *Let $f : [a, b] \to R$ be a mapping with bounded variation on $[a, b]$. Then we have the inequality*

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{2} V_a^b(f) \tag{2.1}$$

where $V_a^b(f)$ denotes the total variation of f on the interval $[a, b]$.
The constant $\frac{1}{2}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^b p(x)df(x) = f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(x)dx \tag{2.2}$$

where

$$p(x) := \begin{cases} x - a \text{ if } x \in [a, \frac{a+b}{2}) \\ x - b \text{ if } x \in [\frac{a+b}{2}, b]. \end{cases}$$

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$.

If $p : [a,b] \to R$ is continuous on $[a,b]$ and $v : [a,b] \to R$ is with bounded variation on $[a,b]$, then

$$\left| \int_a^b p(x)dv(x) \right| = |\lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)})[v(x_{i+1}^{(n)}) - v(x_i^{(n)})]|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})||v(x_{i+1}^{(n)}) - v(x_i^{(n)})|$$

$$\leq \max_{x\in[a,b]} |p(x)| \sup_{\Delta_n} \sum_{i=0}^{n-1} |v(x_{i+1}^{(n)}) - v(x_i^{(n)})| = \max_{x\in[a,b]} |p(x)| V_a^b(v). \tag{2.3}$$

Applying the inequality (2.3) for $p(x)$ as above and $v(x) = f(x), x \in [a,b]$, we get

$$\left| \int_a^b p(x)df(x) \right| \leq \max_{x\in[a,b]} |p(x)| V_a^b(f) = \frac{b-a}{2} V_a^b(f) \tag{2.4}$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq C(b-a)V_a^b(f). \tag{2.5}$$

Consider the mapping $f : [a,b] \to R$,

$$f(x) = \begin{cases} 0 \text{ if } x \in [a,b] \setminus \{\frac{a+b}{2}\} \\ 1 \text{ if } x = \frac{a+b}{2}. \end{cases}$$

Then

$$V_a^b(f) = 2, \int_a^b f(x)\,dx = 0.$$

For this mapping we have

$$\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) = -(b-a)$$

and

$$(b-a)V_a^b(f) = 2(b-a)$$

and then by (2.5) we get

$$b - a \leq 2C(b-a)$$

which implies that $C \geq \frac{1}{2}$ and the sharpness of (2.1) is proved.

The following corollary holds:

COROLLARY 2.2. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) whose derivative is integrable on (a,b). Then we have the inequality:*

$$\left| \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) \right| \leq \frac{1}{2} \parallel f' \parallel_1 (b-a). \tag{2.6}$$

Remark 2.3. It is well known that if $f : [a,b] \to R$ is a convex mapping on $[a,b]$, then *Hermite-Hadamard's* inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f : I \subset R \to R$ is differentiable convex on I and $a,b \in Int(I),\ a < b$, then f' is monotonous nondecreasing on $[a,b]$ and by Theorem 2.1 we get

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\|f'\|_1(b-a) \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

The following corollary for midpoint composite formula holds:

COROLLARY 2.4. *Let $f : [a,b] \to R$ be a mapping with bounded variation on $[a,b]$ and I_h a partition of $[a,b]$. Then we have the midpoint quadrature formula (1.2) and the remainder term $R_M(f,I_h)$ satisfies the estimation:*

$$|R_M(f,I_h)| \leq \frac{1}{2}\gamma(h)V_a^b(f). \tag{2.9}$$

Moreover, the constant $\frac{1}{2}$ is the best possible one, where $\gamma(h) := \max\{h_i | i = 0,\ldots,n-1\}$.

Proof. Applying inequality (2.1) on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ we have

$$\left| \int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right| \leq \frac{1}{2} h_i V_{x_i}^{x_{i+1}}(f).$$

Using the generalized triangle inequality we get

$$|R_M(f,I_h)| = \left| \sum_{i=0}^{n-1} \left(\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right) \right|$$

$$\leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right) h_i \right| \leq \frac{1}{2} L \sum_{i=0}^{n-1} h_i V_{x_i}^{x_{i+1}}(f)$$

$$\leq \frac{1}{2}\gamma(h)\sum_{i=0}^{n-1} V_{x_i}^{x_{i+1}}(f) = \frac{1}{2}\gamma(h)V_a^b(f)$$

and the corollary is proved.

The case of equidistant partitioning is embodied in the following corollary:

COROLLARY 2.5. *Let I_n be an equidistant partitioning of $[a,b]$ and f be as in Theorem 2.1. Then we have the formula (1.6) and the remainder satisfies the estimation:*

$$|R_{M,n}(f)| \leq \frac{1}{2}\cdot\frac{1}{n}(b-a)V_a^b(f). \tag{2.10}$$

Remark 2.6. If we want to approximate the integral $\int_a^b f(x)dx$ by midpoint formula $A_{M,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := \left[\frac{1}{2}\cdot\frac{1}{\varepsilon}(b-a)V_a^b(f)\right] + 1$$

and $[r]$ denotes the integer part of $r \in R$.

Comments 2.7. If the mapping $f : [a,b] \to R$ is neither twice differentiable nor the second derivative is bounded on (a,b), then we can not apply the classical estimation in midpoint formula using the second derivative. But if we assume that f is with bounded variation, then we can use instead the formula (2.9).

We give here a class of mappings which are with bounded variation but having the second derivative unbounded on the given interval.

Let $f_{p,q} : [a,b] \to R$, $f_{p,q}(x) := (x^q - a^q)^p$ where $p \in (1,2)$ and $q \geq 2$. Then obviously

$$f'_{p,q}(x) := pqx^{q-1}(x^q - a^q)^{p-1},\ x \in (a,b)$$

and

$$f''_{p,q}(x) = pq\frac{x^{q-2}[(pq-1)x^q - (q-1)a^q]}{(x^q - a^q)^{2-p}},\ x \in (a,b).$$

It is clear that f is with bounded variation and the total variation is

$$V_a^b(f_{p,q}) = \|f'_{p,q}\|_1(b^q - a^q)^p < \infty$$

but $\lim_{x\to a+} f''_{p,q}(x) = +\infty$.

3. Applications for special means

Let us recall the following means:

1. *Arithmetic mean*

$$A = A(a,b) := \frac{a+b}{2}, \ a,b \geq 0;$$

2. *Geometric mean*

$$G = G(a,b) := \sqrt{ab}, \ a,b \geq 0;$$

3. *Harmonic mean*

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \ a,b > 0;$$

4. *Logarithmic mean*

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, \ a,b > 0, \ a \neq b;$$

5. *Identric mean*

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \ a,b > 0, \ a \neq b;$$

6. *p-Logarithmic mean*

$$L_p = L_p(a,b) := \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \ p \in R\backslash\{-1,0\}, \ a,b > 0, \ a \neq b.$$

It is well known that L_p is monotonous nondecreasing over $p \in R$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A. \tag{3.1}$$

In what follows, by the use of Theorem 2.1, we point out some new inequalities for the above means.

1. Let $f : [a,b] \to R \ (0 < a < b), \ f(x) = x^p, \ p \in R\backslash\{-1,0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L_p(a,b), \ f\left(\frac{a+b}{2}\right) = A^p(a,b),$$

$$\|f'\|_1 = |p|(b-a)L_{p-1}^{p-1}, \ p \in R \setminus \{-1,0,1\}.$$

Using the inequality (2.6) we get

$$|L_p^p(a,b) - A(a^p,b^p)| \leq \frac{|p|}{2}L_{p-1}^{p-1}(b-a)^2. \tag{3.2}$$

2. Let $f : [a,b] \to R \ (0 < a < b), \ f(x) = \frac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L^{-1}(a,b),\ f\left(\frac{a+b}{2}\right) = A^{-1}(a,b),\ \|f'\|_1 = \frac{b-a}{G^2(a,b)}.$$

Using the inequality (2.6) we get

$$0 \le A - L \le \frac{(b-a)^2}{2G^2}LA. \tag{3.3}$$

3. Let $f:[a,b]\to R\ (0<a<b),\ f(x)=\ln x$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = \ln I(a,b),\ f\left(\frac{a+b}{2}\right) = \ln A(a,b),\ \|f'\|_1 = \frac{b-a}{L(a,b)}.$$

Using the inequality (2.6) we get

$$1 \le \frac{A}{I} \le \exp\left[\frac{(b-a)^2}{2L}\right]. \tag{3.4}$$

References

[1] Mitrinović, D.S.; Pečarić, J.E. and Fink, A.M.: *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.