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ELEMENTARY PROOF OF THE REALITY OF THE ZEROS OF β-POLYNOMIALS OF COMPLETE GRAPHS

Xueliang Li[a] **and Ivan Gutman**[b]

[a]*Department of Applied Mathematics, Northwestern Polytechnical University, X'ian, Shaanxi 710072, P. R. China, and*
[b]*Faculty of Science, P. O. Box 60, 34000 Kragujevac, Yugoslavia*

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ABSTRACT. An elementary proof is given for the reality of all zeros of β-polynomials associated with complete graphs.

INTRODUCTION

Let G be a graph on n vertices. The *matching polynomial* of G is defined as [2]:

$$\alpha(G,x) = \sum_{k \geq 0} (-1)^k \, m(G,k) \, x^{n-2k}$$

where $m(G,k)$ is the number of k-matchings of G, i. e., the number of ways in which k mutually non-touching edges are selected in G; $m(G,0) = 1$ and $m(G,1) =$ number of edges of G.

Let C be a circuit contained in the graph G. The subgraph obtained by deleting the vertices of C from G is denoted by $G \setminus C$. The number of vertices of C will be denoted by m. Then $G \setminus C$ possesses $n - m$ vertices.

If C is a Hamiltonian circuit, i. e., if $m = n$ then, by definition, $\alpha(G \setminus C, x) \equiv 1$.

Two graphic polynomials, both denoted by $\beta(G,C,x)$ and related to the matching polynomial, are defined as

$$\beta(G,C,x) = \alpha(G,x) - 2\,\alpha(G \setminus C, x) \tag{1}$$

and

$$\beta(G,C,x) = \alpha(G,x) + 2\,\alpha(G \setminus C, x) \tag{2}$$

For more details on them see [12, 14]. Here we are interested in the following:

Conjecture [4, 5, 7]. *For any circuit C contained in any graph G, all the zeros of $\beta(G,C,x)$, Eqs. (1) and (2), are real.*

Quite a few results have been obtained, corroborating the validity of this conjecture [7, 10, 11, 13, 15], although a complete proof of it is not (yet) known. It was recently demonstrated [12] that the conjecture is true in the case of complete graphs. The proof offered in [12] relies on an earlier published theorem by Turán (from 1958). We now communicate an elementary self–contained proof of the same result.

THE MAIN RESULT

Theorem 1. *Let K_n be the complete graph on n vertices and C any of its circuits. Then all zeros of $\beta(K_n,C,x)$, Eqs. (1), (2), are real.*

Instead of Theorem 1 we demonstrate the validity of a stronger result, namely Theorem 2. In order to state it we need some preparations.

If C is a circuit on m vertices, then $K_n \setminus C = K_{n-m}$, implying that

$$\beta(K_n,C,x) = \alpha(K_n,x) \pm 2\,\alpha(K_{n-m},x) \tag{3}$$

Now, a well–known result from the theory of matching polynomial is [3, 6, 8, 9]:

$$\alpha(K_n,x) = He_n(x) \tag{4}$$

where He_n is one of the standard forms of the Hermite polynomial [1].

Bearing in mind Eqs. (3) and (4) we define a polynomial

$$\beta(n,m,t,x) = He_n(x) + t\,He_{n-m}(x) \tag{5}$$

where $1 \leq m \leq n$ and t is a real number. Clearly, for $n \geq 3$, $|t| = 2$ and $3 \leq m \leq n$, Eq. (5) is the β-polynomial of the complete graph on n vertices, pertaining to a circuit with m vertices.

Theorem 2. *For all (positive integer) values of n, for all $m = 1, 2, \ldots, n$ and for $|t| \leq n-1$ all zeros of the polynomial $\beta(n, m, t, x)$, Eq. (5), are real.*

Obviously, Theorem 1 is a special case of Theorem 2. Therefore in what follows we proceed towards proving Theorem 2. It should be noted that the right–hand side of Eq. (5) is a sort of linear combination of Hermite polynomials.

PREPARATIONS

Some well known properties [1] of the Hermite polynomials are summarized in Lemma 1.

Lemma 1.

(i)

$$He_n(x) = x\, He_{n-1}(x) - (n-1)\, He_{n-2}(x)$$

(ii) all zeros of $He_n(x)$ are real and distinct.

(iii)

$$\frac{d}{dx} He_n(x) = n\, He_{n-1}(x)$$

and hence, $He_n(x)$ has a local extreme x_i if and only if $He_{n-1}(x_i) = 0$. So, the extremes of $He_n(x)$ are distinct.

Throughout this paper $x_1, x_2, \ldots, x_{n-1}$ denote the distinct zeros of $He_{n-1}(x)$.

From Eq. (4) and Theorem 7 of [3], we have

Lemma 2. $|x_i| < 2\sqrt{n-3}$ holds for all $i = 1, 2, \ldots, n-1$.

Lemma 3. *If for all $i = 1, 2, \ldots, n-1$, the sign of $\beta(n, m, t, x_i) = He_n(x_i) + t\, He_{n-m}(x_i)$ is the same as that of $He_n(x_i)$, then all zeros of $\beta(n, m, t, x)$ are real.*

Proof. From Lemma 1 (iii), we have that x_i, $i = 1, 2, \ldots, n-1$ are the extremes of $He_n(x)$. Since $He_n(x)$ does not have multiple zeros (Lemma 1 (ii)), we know that $He_n(x_i) \neq 0$ for all $i = 1, 2, \ldots, n-1$, and that $He_n(x_i)$ and $He_n(x_{i+1})$ have different signs, $i = 1, 2, \ldots, n-2$.

From the definition of $\beta(n, m, t, x)$ and the condition of Lemma 3, we deduce that $\beta(n, m, t, x)$ has at least as many real zeros as $He_n(x)$, that is at least n real zeros. On the other hand the degree of $\beta(n, m, t, x)$ is n. $\square$

Lemma 4. *If* $|He_n(x_i)| > (n-1)\,|He_{n-m}(x_i)|$ *for all* $i = 1, 2, \ldots, n-1$, *then all the zeros of* $\beta(n, m, t, x)$ *are real for* $|t| \le n-1$.

Proof. Since $|He_n(x_i)| > (n-1)\,|He_{n-m}(x_i)| \ge |t|\,|He_{n-m}(x_i)|$ for all $i = 1, 2, \ldots, n-1$, the sign of $\beta(n, m, t, x_i) = He_n(x_i) + t\,He_{n-m}(x_i)$ depends only on the sign of $He_n(x_i)$. Lemma 4 follows from Lemma 3. $\square$

PROOF OF THEOREM 2

Bearing in mind that $He_{n-1}(x_i) = 0$, from Lemma 4 we immediately get

Lemma 5. *All zeros of the polynomial* $\beta(n, 1, t, x)$ *are real for* $n \ge 1$ *and any real value of the parameter* t.

Lemma 5 implies the validity of Theorem 2 for $m = 1$. What remains is to consider the case $m \ge 2$. Therefore, in what follows it will be assumed that $2 \le m \le n$.

Define the auxiliary quantities $a_{n,m}$ as

$$a_{n,m} = \max\left\{ \frac{|He_{n-m}(x_i)|}{|He_n(x_i)|} \;\middle|\; i = 1, 2, \cdots, n-1 \right\} \tag{6}$$

Because of Lemma 4, if

$$a_{n,m} < \frac{1}{n-1} \tag{7}$$

then all the zeros of $\beta(n, m, t, x)$ are real for $|t| \le n-1$. Therefore, in order to complete the proof of Theorem 2 we only need to verify the inequality (7).

From Lemma 1 (i),

$$He_{n-m}(x) = \frac{x\,He_{n-m+1}(x) - He_{n-m+2}(x)}{n-m+1} \tag{8}$$

which, combined with Lemma 2 yields

$$\begin{aligned}
a_{n,2} &= \frac{1}{n-1} \\
a_{n,3} &< \frac{2\sqrt{n-3}\,a_{n,2}}{n-2} = \frac{2\sqrt{n-3}}{(n-1)(n-2)} \\
a_{n,4} &< \frac{1}{n-3}\left(2\sqrt{n-3}\,a_{n,3} + a_{n,2}\right) \\
&\le \frac{1}{n-3}\left[(2\sqrt{n-3})^2\,\frac{1}{(n-1)(n-2)} + \frac{1}{n-1}\right]
\end{aligned} \tag{9}$$

$$= \frac{(2\sqrt{n-3})^2}{(n-1)(n-2)(n-3)} \left[1 + \frac{n-2}{4(n-3)}\right] \tag{10}$$

Lemma 6. *Let* $b_{n,m}$ *be defined as*

$$b_{n,m} = \left[(1+\sqrt{2})(\sqrt{n-3})\right]^{m-2} \frac{(n-m)!}{(n-1)!} \tag{11}$$

Then

$$a_{n,m} < b_{n,m} \tag{12}$$

holds for $n \geq 6$ *and* $m \geq 3$.

Proof proceeds by induction on m. For $m = 3$, relation (12) follows from (9). For $m = 4$, since $n \geq 6$, relation (12) follows from (10).

Suppose inequality (12) holds for $m-1$ and $m-2$. Then for any $m \geq 5$, by using Eq. (8), Lemma 2 and the induction hypothesis we get

$$\begin{aligned}
a_{n,m} &< \frac{1}{n-m+1}\left(2\sqrt{n-3}\,a_{n,m-1} + a_{n,m-2}\right) \\
&\leq \frac{1}{n-m+1}\Bigg[2\sqrt{n-3}\frac{[(1+\sqrt{2})(\sqrt{n-3})]^{m-3}\,(n-m+1)!}{(n-1)!} + \\
&\qquad \frac{[(1+\sqrt{2})(\sqrt{n-3})]^{m-4}\,(n-m+2)!]}{(n-1)!}\Bigg] \\
&= \frac{(n-m)!\,(1+\sqrt{2})^{m-2}\,(\sqrt{n-3})^{m-2}}{(n-1)!}\left[\frac{2}{1+\sqrt{2}} + \frac{n-m+2}{(1+\sqrt{2})^2(\sqrt{n-3})^2}\right] \\
&= b_{n,m}\left[\frac{2}{1+\sqrt{2}} + \frac{n-m+2}{(1+\sqrt{2})^2(\sqrt{n-3})^2}\right]
\end{aligned}$$

Since $m \geq 5$,

$$\frac{2}{1+\sqrt{2}} + \frac{n-m+2}{(1+\sqrt{2})^2\,(\sqrt{n-3})^2} \leq \frac{2}{1+\sqrt{2}} + \frac{1}{(1+\sqrt{2})^2} = 1$$

and we arrive at our inequality (12). □

Lemma 7.

(i) If $m \leq (n+1) - (1+\sqrt{2})\sqrt{n-3}$, *then* $b_{n,m}$ *is monotonically decreasing on* m.

(ii) If $m \geq (n+1) - (1+\sqrt{2})\sqrt{n-3}$, *then* $b_{n,m}$ *is monotonically increasing on* m.

Proof. Consider the ratio $b_{n,m}/b_{n,m-1}$. □

As a consequence of Lemma 7, we have

$$\begin{cases} b_{n,m} & \le & \max\{b_{n,2}, b_{n,n}\} \\ \\ b_{n,m} & < & 1/(n-1) \quad \text{for all } 3 \le m \le (n+1) - (1+\sqrt{2})\sqrt{n-3} \end{cases} \tag{13}$$

Recall that as a special case of Eq. (11),

$$b_{n,2} = \frac{1}{n-1} \tag{14}$$

In view of relations (13) and (14), in order to show that $b_{n,m} < 1/(n-1)$ for $m > (n+1) - (1+\sqrt{2})\sqrt{n-3}$, it is sufficient to prove $b_{n,n} < 1/(n-1)$. Denote for brevity, $c_n = (n-1)\, b_{n,n}$. We thus need to show that $c_n < 1$.

Lemma 8. *For* $n \ge 17$, c_n *is monotonically decreasing on* n.

Proof. Consider

$$\begin{aligned} \frac{c_{n+1}}{c_n} &= \frac{(1+\sqrt{2})^{n-1}\,(\sqrt{n-2})^{n-1}}{(n-1)!} \cdot \frac{(n-2)!}{(1+\sqrt{2})^{n-2}\,(\sqrt{n-3})^{n-2}} \\ &= \frac{(1+\sqrt{2})\sqrt{n-3}}{n-1} \left(\sqrt{\frac{n-2}{n-3}}\right)^{n-1} \\ &= \frac{1+\sqrt{2}}{\sqrt{n-3}} \cdot \frac{n-2}{n-1} \left(1 + \frac{1}{n-3}\right)^{(n-3)/2} \\ &< \frac{1+\sqrt{2}}{\sqrt{n-3}}\, e^{1/2} \end{aligned}$$

Because

$$(1+\sqrt{2})\, e^{1/2} \approx 3.993 < 4$$

we see that $c_{n+1} < c_n$ whenever $n - 3 \ge 4^2$, i. e., $n \ge 19$.

The fact that c_n monotonically decreases already from $n = 17$ is checked by direct calculation: $c_{16} = 164.4...$, $c_{17} = 166.3...$, $c_{18} = 163.1...$, $c_{19} = 155.2...$. □

Lemma 9. *If* $n \ge 39$, *then* $c_n < 1$.

Because $c_{39} = 0.65...$, Lemma 9 follows from Lemma 8. □

Note that the bound 39 in Lemma 9 cannot be lowered, since $c_{38} = 1.006...$.

Lemma 9 is tantamount to

Lemma 10. *If* $n \ge 39$, *then* $b_{n,m} < 1/(n-1)$ *for any* $m = 3, \ldots, n$. □

Lemma 11.

(i) If $n \geq 39$ *and* $|t| \leq n-1$, *then ann the zeros of* $\beta(n,m,t,x)$ *are real.*

(ii) If $3 \leq m \leq (n+1)-(1+\sqrt{2})\sqrt{n-3}$ *and* $|t| \leq n-1$, *then all the zeros of* $\beta(n,m,t,x)$ *are real.*

Proof. For $n \leq 5$, one can easily check that all the zeros of $\beta(n,m,t,x)$ are real. Therefore in the following we assume that $n \geq 6$.

From Lemmas 10 and 6 as well as Eq. (13) we know, under the condition of Lemma 11, that $a_{n,m} < 1/(n-1)$. Bearing in mind Eq. (6), we conclude that $|He_n(x_i)| > (n-1)\,|He_{n-m}(x_i)|$ for all $i = 1, 2, \ldots, n-1$. Lemma 11 follows then from Lemma 4. $\square$

By means of Lemmas 5 and 11, Theorem 2 has been verified for all n and m, except for $6 \leq n \leq 38$ and $(n+1)-(1+\sqrt{2})\sqrt{n-3} < m \leq n$, i. e., except for a finite number of cases. The checking that also in these remaining few cases all the zeros of all the β-polynomials are real for $|t| \leq n-1$ has been done by direct (tedious, yet elementary) calculations. Their details are omitted.

By this, the proof of Theorem 2 has been completed.

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