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ON PERMUTATIONS IN BIPARTITE GRAPHS

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ABSTRACT. Let G be a graph whose vertices are labeled by $1, 2, \ldots, n$. A permutation $P : [1, 2, \ldots, n] \to [P(1), P(2), \ldots, P(n)]$ is said to be a graphic permutation of G if the vertices i and $P(i)$ of G are adjacent for all $i = 1, 2, \ldots, n$. It is shown that a bipartite graph with m perfect matchings has m^2 graphic permutations. Some consequences of this result on the determinant of the adjacency matrix of a bipartite graph are pointed out.

INTRODUCTION

In this paper we consider finite bipartite graphs without loops and multiple edges. Let G be such a graph and let its vertices be labeled by $1, 2, \ldots, n$.

It is well known that bipartite graphs do not possess odd–membered circuits and that their vertices can be colored by two colors (say, black and white), so that all first neighbors of a white vertex are black, and vice versa.

As an example, in Fig. 1 we show the numbering and the coloring of the vertices of a bipartite graph G_0.

A perfect matching of a graph G is a set of independent edges of G that cover all the vertices of G [7]. If G is bipartite then any edge in any of its perfect matchings necessarily connects a black and a white vertex.

Three perfect matchings of the graph G_0 from Fig. 1 are represented by the diagrams M_1, M_2 and M_3 in Fig. 2; the edges belonging to the perfect matchings are indicated by heavy lines.

Fig. 1.

The number of perfect matchings of a graph G is denoted by $m = m(G)$. For instance, $m(G_0) = 6$.

Let

$$P = \begin{pmatrix} 1 & 2 & \dots & n \\ P(1) & P(2) & \dots & P(n) \end{pmatrix} \tag{1}$$

be a permutation of the numbers $1, 2, \dots, n$. We say that P is a graphic permutation of G, or that G contains the permutation P, if all the vertex pairs $(i\,,\,P(i))$; $i = 1, 2, \dots, n$, are adjacent in G.

Without loss of generality, a graph G (with undirected edges) can always be regarded as a digraph in which every pair of adjacent vertices is connected by two oppositely directed arcs. If so, then a graphic permutation can be understood as a spanning sub–digraph of G, in which exactly one arc starts from every vertex and exactly one arc ends at every vertex.

In Fig. 3 are presented the digraphs corresponding to the permutations

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & 4 & 5 & 6 & 16 & 8 & 7 & 10 & 9 & 12 & 11 & 14 & 1 & 13 & 15 \end{pmatrix}$$

Fig. 2.

and

$$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 3 & 4 & 5 & 6 & 16 & 8 & 9 & 10 & 7 & 12 & 11 & 14 & 1 & 13 & 15 \end{pmatrix}$$

We immediately see that P_1 is, whereas P_2 is not a graphic permutation of G_0 (because the vertices 7 and 10 are not adjacent in G_0, cf. Fig. 1).

THE NUMBER OF GRAPHIC PERMUTATIONS IN A BIPARTITE GRAPH

Theorem 1. *The number of graphic permutations in a bipartite graph G is equal to m^2, where m is the number of perfect matchings of G.*

Proof. Color the vertices of G by two colors (black and white), so that adjacent vertices are colored differently. Suppose first that G has at least one perfect matching, i. e., that $m \geq 1$.

Fig. 3.

Let $M_1, \ldots, M_m$ be the perfect matchings of G. Then every edge in M_i, $i = 1, \ldots, m$, connects a black and a white vertex.

For $i = 1, \ldots, m$ construct the digraph $M_i^\bullet$ by directing the edges of M_i so that they start at black vertices of G. Construct the digraph M_i° by directing the edges of M_i so that they start at white vertices of G. Note that both M_i, $M_i^\bullet$ and M_i° have the same vertex sets, coinciding with the vertex set of G.

Examples of the digraphs $M_i^\bullet$ and M_i° of G_0 can be found in Fig. 2; their arcs are directed according to the coloring indicated in Fig. 1.

Construct now a digraph $R_{ij} = M_i^\bullet \cup M_j^\circ$ so that it has the same vertex set as $M_i^\bullet$ and M_j° (i. e., as G), whereas the edge set of R_{ij} is the union of the edge sets of $M_i^\bullet$ and M_j°.

Evidently, for a fixed labeling of the vertices of G, two graphs R_{ab} and R_{cd} are isomorphic only if $M_a^\bullet = M_c^\bullet$ and $M_b^\circ = M_d^\circ$. Consequently, there are m^2 labeled digraphs R_{ij}.

In order to arrive at Theorem 1 it is now sufficient to verify that there exists a one–to–one correspondence between the digraphs R_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, m$, and the graphic permutations of G.

Indeed, because one arc starts from, and ends at every vertex of R_{ij} we have:

Lemma 1. *Every digraph R_{ij} represents a graphic permutation of G.*

Further, every graphic permutation of a bipartite graph can be decomposed into two components. One component is formed by the arcs that start from black vertices, the other by the arcs that start from white vertices. Bearing in mind the definition of a permutation, we see that this decomposition results in an $M_i^\bullet$– and an M_j°–digraph. Hence we arrived at:

Lemma 2. *Every graphic permutation of G corresponds to a unique digraph of the type R_{ij}.*

For instance, the permutation P_1 from Fig. 3 (which is graphic with respect to G_0), coincides with the digraph $R_{12} = M_1^\bullet \cup M_2^\circ$, where $M_1^\bullet$ and M_2° are given in Fig. 2.

Lemmas 1 and 2 straightforwardly lead to Theorem 1, provided $m \geq 1$. It, therefore, remains to show that Theorem 1 holds also if $m = 0$. This, however, follows from the above reasoning: the existence of a graphic permutation (in a bipartite graph) implies the existence of at least one R_{ij}–digraph, i. e., the existence of at least one $M_i^\bullet$– and at least one M_j°–digraph, i. e., the existence of at least one perfect matching.

Hence, if $m = 0$, then the graph G cannot contain permutations and Theorem 1 is satisfied. $\square$

Remark 1. Theorem 1 holds for an arbitrary labeling and an arbitrary coloring of the vertices of G. In particular, it holds for disconnected graphs (for which the coloring of the vertices is not unique).

Remark 2. Theorem 1 cannot be extended to non–bipartite graphs. One, however, has a weaker result:

Corollary 1.1. *The number of graphic permutations in a non-bipartite graph H is greater than or equal to $m(H)^2$.*

We wish here to emphasize the following special case of Theorem 1:

Corollary 1.2. *A bipartite graph contains graphic permutations if and only if it possesses perfect matchings.*

SOME FURTHER COROLLARIES OF THEOREM 1

The adjacency matrix A of a (labeled) n-vertex graph G is defined as the square matrix of order n, whose elements are given by

$$A_{ij} = \begin{cases} 1 & \text{if the vertices } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise .} \end{cases} \tag{2}$$

The determinant of the adjacency matrix is thus equal to

$$\det A = \sum_P (-1)^{\pi(P)} A_{1,P(1)} A_{2,P(2)} \cdots A_{n,P(n)} \tag{3}$$

where P is a permutation (see Eq. (1)), $\pi(P)$ is its parity and the summation goes over all $n!$ permutations.

Bearing in mind Eq. (2) we see that the term $A_{1,P(1)} A_{2,P(2)} \cdots A_{n,P(n)}$ will be non–zero if and only if $A_{1,P(1)} = A_{2,P(2)} = \cdots = A_{n,P(n)} = 1$, i. e., if and only if all the vertex pairs $(i\,,\,P(i))\,;\,i=1,2,\ldots,n\,,$ are adjacent in $G\,.$ This, on the other hand, is just the definition of a graphic permutation. Hence, from (2) and (3) follows that

$$\det A = {\sum_P}' \,(-1)^{\pi(P)} \tag{4}$$

where the summation now embraces only graphic permutations.

As well known, the parity of a permutation P depends on its cycle–decomposition and is equal to the parity of the number of its even–membered cycles. Denote by $c_i(P)$ the number of i-membered cycles contained in the permutation $P\,.$ Introduce, further, the quantities e and f :

$$e(P) = \sum_{i\geq 1} c_{2i}(P)$$

and

$$f(P) = \sum_{i\geq 1} c_{4i}(P)\ .$$

Thus $e(P)$ counts the even–membered cycles in P whereas $f(P)$ is equal to the number of cycles in P whose sizes are divisible by four. Because of $\pi(P)\equiv e(P)\pmod 2$, we may rewrite Eq. (4) as

$$\det A = {\sum_P}'(-1)^{e(P)}\ . \tag{5}$$

Formula (5) was first time obtained by Harary [5]; it holds for arbitrary graphs and digraphs.

If G is a bipartite graph then Eq. (5) can be somewhat modified. Then for all graphic permutations $P\,,\ \ c_i(P)=0$ if i is odd, and we have

$$\sum_{i\geq 1} 2i\, c_{2i}(P) = n\ .$$

Consequently,

$$\begin{aligned}
n/2 + f(P) &= \sum_{i\geq 1} i\, c_{2i}(P) + \sum_{i\geq 1} c_{4i}(P) \\
&= \left[\sum_{i\geq 1} 2i c_{4i}(P) + \sum_{i\geq 0} (2i+1)\, c_{4i+2}(P)\right] + \sum_{i\geq 1} c_{4i}(P) \\
&= \sum_{i\geq 1} (2i+1)\, c_{4i}(P) + \sum_{i\geq 0} (2i+1)\, c_{4i+2}(P) \\
&\equiv \left[\sum_{i\geq 1} c_{4i}(P) + \sum_{i\geq 0} c_{4i+2}(P)\right] \pmod 2
\end{aligned}$$

i. e.,

$$n/2 + f(P) \equiv e(P) \pmod 2 \ . \tag{6}$$

Substituting (6) back into (5) we obtain

Lemma 3. *For a bipartite graph with n vertices,*

$$\det A = (-1)^{n/2} \sum_P{}' (-1)^{f(P)} \ . \tag{7}$$

Note that when n is odd, then G cannot contain graphic permutations and, consequently, $\det A = 0$.

The advantage of formula (7) relative to (5) is seen from the following observation. If all graphic permutations of G satisfy the condition $f(P) = 0$, then the sum on the right–hand side of (7) is equal to the number of graphic permutations of G, which, on the other hand, is determined by Theorem 1. The following two corollaries are now obvious.

Corollary 1.3. *If a bipartite graph possesses no circuit whose size is divisible by 4, then*

$$\det A = (-1)^{n/2}\, m^2 \ . \tag{8}$$

Corollary 1.4. *For a forest (= acyclic graph) F,*

$$\det A = \begin{cases} (-1)^{n/2} & \text{if } F \text{ has a perfect matching} \\ 0 & \text{if } F \text{ has no perfect matching} \ . \end{cases}$$

Forests, of course, can have at most one perfect matching [7]. The result stated here as Corollary 1.4 is well known in graph spectral theory (see [1], p. 37).

ONE MORE COROLLARY: APPLICATION OF THEOREM 1 TO HEXAGONAL SYSTEMS

In addition to the graphs specified in Corollary 1.3, there exist other non–trivial cases for which $f(P) = 0$ holds for all graphic permutations. One distinguished class of graphs having this property are the so–called hexagonal systems.

According to Sachs [8], hexagonal systems are defined as follows. A hexagonal unit cell is a plane region bounded by a regular hexagon of side length 1. A hexagonal system is then a finite connected plane graph with no cut–vertices, in which every interior region is a hexagonal unit cell.

The graph G_0 depicted in Fig. 1 can serve as an example of a hexagonal system. For more details on hexagonal systems and their applications in chemistry see [3, 4, 6, 7].

The vertices of a hexagonal system can be divided into internal and external. A vertex is said to be internal if it belongs to three hexagonal unit cells; otherwise it is external. The external vertices form the boundary of the respective hexagonal system.

For instance, the vertices 15 and 16 of G_0 are internal whereas the vertices $1, 2, \ldots, 14$ are external (see Fig. 1).

Let n, n_i and h be the numbers of vertices, internal vertices and hexagonal unit cells, respectively, of a hexagonal system. Then the following identity is easily proved by induction on h [4, 6]:

$$n = 4h + 2 - n_i \ . \tag{9}$$

Rewriting (9) as

$$n_i = 2h + 1 - \frac{1}{2}\,(n - n_i)$$

and bearing in mind that $n - n_i$ is just the size of the boundary, we see that in the interior of a boundary whose size is divisible by four (in which case $(n - n_i)/2$ is an even integer), there is an odd number of vertices. This can be readily generalized as:

Lemma 4. *Let H be a hexagonal system and γ its circuit whose size is divisible by four. Then in the interior of γ there is an odd number of vertices of H.*

An immediate consequence of Lemma 4 is that graphic permutations of hexagonal systems cannot contain cycles whose sizes are divisible by four. To see this observe that in every graphic permutation P the vertices lying in the interior of γ are mapped either on each other or on the vertices of γ. If γ would correspond to a cycle of P, then the internal vertices would have to be mapped only on each other. Because their number is odd, this would be possible only if at least one odd–membered cycle is present in P. But because hexagonal systems are bipartite graphs, their graphic permutations must not possess odd–membered cycles. Hence γ cannot induce a cycle in P.

Combining this property of hexagonal systems with Theorem 1 and Lemma 3, we obtain:

Corollary 1.5. *Hexagonal systems obey Eq. (8).*

The result stated here as Corollary 1.5 was reported in 1952 by Dewar and Longuet–Higgins [2]. Its rigorous proof was, however, published only in 1980 (see [1], p. 243), using a way of reasoning that differs from ours.

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