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SOME GENERALIZATIONS OF ONE MITRINOVIĆ'S PROBLEM

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ABSTRACT. It is shown that, if y_0 is the solution of the differential equation (1), then y_0' is the solution of the differential equation (1.1).

Besides this, it is shown that general solution of differential equation (4) and (5) satisfy the relation (6), if their coefficients are related by the relations (10) and (11).

D. S. Mitrinović [1] define the following problem:

If y_0 is the solution of the differential equation

$$xy'' = y' + xy = 0$$

then y_0' is the solution of the differential equation

$$x^2y'' + 2xy' + (x^2 - 1)y = 0.$$

Prove this statement.

This result is indicative because it initiates the idea to find the answer to the following more general question:

If y_0 is the solution of the differential equation

$$a_0y'' + a_1y' + a_2y = 0, \quad a_0' \neq 0, \tag{1}$$

which differential equation has the solution y_0'?

The simplest way to give the answer to the above question is to write equation (1) in the form

$$y''(a_1/a_0)y' - (a_2/a_0)y = 0$$

and then to find its derivative. After this operation the following equation is obtained

$$y''' + (a_1/a_0)y'' + [(a_1/a_0)' + (a_2/a_0)]y' + (a_2/a_0)'y = 0. \tag{2}$$

It is evident that the differential equation (2) will have the solution y_0', if and only if the condition

$$(a_2/a_0)' = 0 \Longrightarrow a_2 = a_0, \tag{3}$$

is satisfied, so that, whit (2) and (3), the sought differential equation has the form

$$y'' + (a_1/a_0)y' + [(a_1/a_0)' + 1]y = 0$$

or

$$a_0^2 y'' + a_1 a_0 y' + (a_1' a_0 - a_1 a_0' + a_0^2)y = 0.$$

Therefore, it is proved the following

Theorem 1. *If y_0 is the solution of the solution of the differential equation* (1), *then y_0' is the solution of the differential equation* (1.1).

However, regarding the above presentation, the answer to the following question may be sought:

Which relation must exist between the coefficients of the equation

$$y'' + a_1(x)y' + a_2(x)y = 0 \tag{4}$$

and

$$z'' + b_1(x)z' + b_2(x)z = 0 \tag{5}$$

under the condition that their general solutions Y and Z satisfy relation

$$Z = Y'. \tag{6}$$

In order to answer the above question, it should be noted that, taking into account (6)

$$Y'' + a_1 Y' + a_2 Y \equiv 0 \tag{4.1}$$

and

$$Y''' + b_1 Y'' + b_2 Y' \equiv 0. \tag{5.1}$$

Besides this, when (4.1) is differentiated, the following identity is obtained

$$Y''' + a_1 Y'' + (a_1' + a_2)Y' + a_2' Y \equiv 0,$$

and, after subtracting (5.1), the identity

$$(a_1 - b_1)Y'' + (a_1' + a_2 - b_2)Y' + a_2' Y \equiv 0 \tag{7}$$

is obtained.

If Y'' is calculated from (4.1) and the value

$$Y'' \equiv -a_1 Y' - a_2 Y$$

is substituted into the identity (7), the following identity is obtained

$$(-a_1^2 + a_1 b_1 + a_1' + a_2 - b_2)Y' + (-a_1 a_2 + a_2 b_1 + a_2')Y = 0$$

which is satisfied for any x, if

$$-a_1^2 + a_1 b_1 + a_1' + a_2 - b_2 = 0 \tag{8}$$

and

$$-a_1 a_2 + a_2 b_1 + a_2' = 0. \tag{9}$$

From equation (9) follows

$$b_1 = (a_1 a_2 - a_2')/a_2 \Longleftrightarrow b_1 = a_1 - (\ln|a_2|)'\,. \tag{10}$$

If the value (10) is substituted into equation (8), the following equation is obtained

$$a_1 a_2' - a_1' a_2 - a_2^2 + a_2 b_2 = 0,$$

which, after division by a_2^2, can be written in the form

$$b_2/a_2 - (a_1/a_2)' - 1 = 0 \Longleftrightarrow b_2 = [(a_1/a_2)' + 1]a_2. \tag{11}$$

Therefore, from the above follows that it is proven the following

Theorem 2. *The general solutions of differential equations* (4) *and* (5) *are related by* (6)*, if and only if their coefficients satisfy the conditions* (10) *and* (11)*.*

CONCLUSION

It is obvious that the results expressed by theorem 1 and 2 are of interest for the analytical theory of differential equations, because they provide an extension of the integrable differential equations.

In addition to this statement is the equivalency

$$(a_2/a_0)' = 0 \Longleftrightarrow a_2 = Ca_0$$

from which follows that, from the integrability of the differential equation (1.1), follows integrability of the differential equation

$$a_0^2 y'' + a_1 a_0 y' + (a_1' a_0 - a_1 a_0' + Ca_0^2)y = 0.$$

Finally, it should be pointed out that the results expressed by theorem 1 and 2 can simply be generalized, so that they are valid for the corresponding differential equations of the n-th order.

References

[1] D. S. Mitrinović, *Zbornik matematičkih problema IV*, Naučna knjiga, Beograd 1972.