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AN GENERALIZATION OF THE RIESZ POTENTIAL

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ABSTRACT. In this paper, we study L_p $(p \geq 1)$ continuity of the function

$$\big(\forall x \in R^n\big)\big(A_\varphi f\big)(x) := \int_{R^n} K(x,y) f(y) dy$$

where $f : R^n \to R$ and $K : R^n \times R^n \to R$.

In this paper we study continuity of the function

$$\big(\forall x \in R^n\big)\big(A_\varphi f\big)(x) := \int_{R^n} K(x,y) f(y) dy, \tag{1}$$

where $f : R^n \to R$, function $K : R^n \times R^n \to R$ is defined by

$$\big(\forall (x,y) \in R^n \times R^n\big) K(x,y) := \frac{1}{\varphi(||x-y||)}, \tag{2}$$

$||\cdot||$ is the Euclidean norm in R^n, function $\varphi : R_0^+ \to R_0^+$ $(R_0^+ := [0, +\infty[)$ has the finite derivative and the following is valid:

$$\varphi(0) = 0, \tag{3}$$

$$\frac{\varphi'(t)}{\varphi(t)} \searrow \text{on} \quad]0, \Delta[, \quad (\Delta > 0), \tag{4}$$

$$\frac{t \cdot \varphi'(t)}{\varphi(t)} \sim c > 0, \quad (t \to +0), \tag{5}$$

$$\int_0^r \frac{t^{n-1}}{\varphi^p(t)} dt = \mathcal{O}\left(\frac{r^n}{\varphi^p(r)}\right), \quad (r \to +0). \tag{6}$$

The function $A_\varphi f$, when

$$\big(\forall t \in [0,+\infty[\big)\varphi(t) := t^{n-a}, \quad (a < n), \tag{7}$$

is called the Riesz potential of function f. When φ satisfies the conditions (3), (4), (5) and (6), $A_\varphi f$ is the generalization of the Riesz potential of the function f.

The L_p continuity, or continuity in the L_p norm, of a real or a complex function g at point $x \in R^n$ thus:

$$\left\{\frac{1}{mB_r}\int_{B_r} |\, g(x+s) - g(x)\,|^p\, ds\right\}^{\frac{1}{p}} \to 0, \quad (r \to +0), \tag{8}$$

where mB_r is the volume of a ball of radius r with center at $O \in R^n$ [3, page 65].

In this paper is proved that, depending on conditions imposed on f and φ, the function $A_\varphi f$ is L_p continuous at every point at which it exist.

Theorem 1. *If $p \geq 1$ and $A_\varphi f$ exists at point $x \in R^n$, then $A_\varphi f$ is L_p continuous at x.* □

Proof. It is sufficient to consider the case when $x = 0$ and to prove that ([1] and [2])

$$\left\{\int_{||s||<r} \big|\big(A_\varphi f\big)(s) - \big(A_\varphi f\big)(0)\big|^p\, ds\right\}^{\frac{1}{p}} = o\big(r^{\frac{n}{p}}\big) \tag{9}$$

holds when $r \to +0$. From (9), (1) and (2) follows:

$$\begin{aligned}
&\left\{\int_{||s||<r} \big|\big(A_\varphi f\big)(s) - \big(A_\varphi f\big)(0)\big|^p\, ds\right\}^{\frac{1}{p}} \qquad (10)\\
&= \left\{\int_{||s||<r} \left|\int_{R^n} \left[\frac{1}{\varphi(||s-y||)} - \frac{1}{\varphi(||-y||)}\right] f(y)dy\right|^p ds\right\}^{\frac{1}{p}}\\
&\leq \left\{\int_{||s||<r} \left[\int_{||y||<2r} \left|\frac{1}{\varphi(||s-y||)} - \frac{1}{\varphi(||y||)}\right| |f(y)|dy\right]^p ds\right\}^{\frac{1}{p}}\\
&+ \left\{\int_{||s||<r} \left[\int_{||y||>2r} \left|\frac{1}{\varphi(||s-y||)} - \frac{1}{\varphi(||y||)}\right| |f(y)|dy\right]^p ds\right\}^{\frac{1}{p}}\\
&\leq \left\{\int_{||s||<r} \left[\int_{||y||<2r} \frac{|f(y)|}{\varphi(||s-y||)} dy\right]^p ds\right\}^{\frac{1}{p}}\\
&+ \left\{\int_{||s||<r} \left[\int_{||y||<2r} \frac{|f(y)|}{\varphi(||y||)} dy\right]^p ds\right\}^{\frac{1}{p}}\\
&+ \left\{\int_{||s||<r} \left[\int_{||y||>2r} \left|\frac{1}{\varphi(||s-y||)} - \frac{1}{\varphi(||y||)}\right| |f(y)|dy\right]^p ds\right\}^{\frac{1}{p}}\\
&= S_1 + S_2 + S_3
\end{aligned}$$

$$S_1 \leq \int_{||y||<2r} \left[\int_{||s||<r} \frac{|f(y)|^p}{\varphi^p(||y-s||)} ds \right]^{\frac{1}{p}} dy \tag{11}$$
$$\leq \int_{||y||<2r} |f(y)| \left[\int_{||y-s||<3r} \frac{ds}{\varphi^p(||y-s||)} \right]^{\frac{1}{p}} dy$$
$$= \int_{||y||<2r} |f(y)| \left[n \cdot mB_1 \cdot \int_0^{3r} \frac{t^{n-1}}{\varphi^p(t)} dt \right]^{\frac{1}{p}} dy$$
$$= \mathcal{O}\Bigg(\frac{(3r)^{\frac{n}{p}}}{\varphi(3r)} \cdot \varphi(2r) \int_{||y||<2r} \frac{|f(y)|}{\varphi(||y||)} dy \Bigg) = o\big(r^{\frac{n}{p}}\big) \quad (r \to +0,)$$

since the following is valid:

$$\int_{||x||<r} h\,(||x||)\, dx = n \cdot mB_1 \cdot \int_0^r t^{n-1}\, h(t)\, dt, \tag{12}$$

if $t^{n-1}h(t)$ is non–negative and measurable or integrable function at $]0, r[$ (where mB_1 denotes the volume of the ball of radius 1 with center at $0 \in R^n$).

Inside integrals in S_2 is the same as integral in

$$\mathcal{O}\Bigg(\frac{(3r)^{\frac{n}{p}}}{\varphi(3r)} \cdot \varphi(2r) \int_{||y||<2r} \frac{|f(y)|}{\varphi(||y||)} dy \Bigg) \tag{13}$$

and it tends to zero when $r \to +0$. This is why:

$$S_2 = o\Bigg(\int_{||s||<r} ds \Bigg)^{\frac{1}{p}} = o\big(r^{\frac{n}{p}}\big), \tag{14}$$

when $r \to +0$.

Using the Minkowski inequality for integrals and (12), we obtain:

$$S_3 = \mathcal{O}\Bigg\{ \int_{||s||<r} \left[\int_{||y||>2r} \frac{\varphi'(||y||)}{\varphi^2(||y||)} ||s|| |f(y)| dy \right]^p ds \Bigg\}^{\frac{1}{p}} \tag{15}$$
$$= \mathcal{O}\Bigg\{ \int_{||y||>2r} \Bigg\{ \int_{||s||<r} \left[\frac{\varphi'(||y||)}{\varphi^2(||y||)} \right]^p ||s||^p |f(y)|^p ds \Bigg\}^{\frac{1}{p}} dy \Bigg\}$$
$$= \mathcal{O}\Bigg\{ \int_{||y||>2r} \frac{\varphi'(||y||)|f(y)|}{\varphi(||y||)\varphi(||y||)} dy \Big(\int_{||s||<r} ||s||^p ds \Big)^{\frac{1}{p}} \Bigg\}$$
$$= \mathcal{O}\Bigg\{ \int_{||y||>2r} \frac{\varphi'(||y||)|f(y)|}{\varphi(||y||)\varphi(||y||)} dy \Big(\int_0^r t^{n-1+p} dt \Big)^{\frac{1}{p}} \Bigg\}$$
$$= \mathcal{O}\Bigg\{ r^{\frac{n}{p}+1} \int_{||y||>2r} \frac{\varphi'(||y||)|f(y)|}{\varphi(||y||)\varphi(||y||)} dy \Bigg\}.$$

For $0 < r < \min\{1, \Delta\}$, from (4) and (5) follows:

$$\begin{aligned}
&\int_{||y||>2r} \frac{\varphi'(||y||)}{\varphi(||y||)} \cdot \frac{|f(y)|}{\varphi(||y||)} dy \qquad (16)\\
&= \left(\int_{||y||>2\sqrt{r}} + \int_{2\sqrt{r}\geq||y||>r} \right) \frac{\varphi'(||y||)}{\varphi(||y||)} \cdot \frac{|f(y)|}{\varphi(||y||)} dy \\
&\leq \frac{\varphi'(2\sqrt{r})}{\varphi(2\sqrt{r})} \int_{||y||>2\sqrt{r}} \frac{|f(y)|}{\varphi(||y||)} dy + \frac{\varphi'(2r)}{\varphi(2r)} \int_{2\sqrt{r}\geq||y||>r} \frac{|f(y)|}{\varphi(||y||)} dy \\
&\leq \frac{\varphi'(2\sqrt{r})}{\varphi(2\sqrt{r})} \int_{R^n} \frac{|f(y)|}{\varphi(||y||)} dy + \frac{\varphi'(2r)}{\varphi(2r)} \int_{||y||\leq 2\sqrt{r}} \frac{|f(y)|}{\varphi(||y||)} dy \\
&= \mathcal{O}\left(\frac{1}{2\sqrt{r}}\right) + o\left(\frac{1}{r}\right) \\
&= o\left(\frac{1}{r}\right).
\end{aligned}$$

From (15) and (16) follows:

$$S_3 = o\big(r^{\frac{n}{p}}\big) \quad (r \to +0). \tag{17}$$

From (10), (11), (14) and (17) follows (9). ■

Specially, if φ is defined by (7), i.e. if $A_\varphi f$ is a Riesz potential of f, then the conditions (3), (4), (5) and (6) are valid if

$$1 - \frac{1}{p} < \frac{a}{n} < 1,$$

and if $A_\varphi f$ exists at $x \in R^n$, then $A_\varphi f$ is L_p continuous at x. From there and from Theorem 1 follows the known [1]:

Theorem 2. *If $f : R^n \to C, \alpha \in C$,*

$$\big(\forall x \in R^n\big)\big(A^\alpha f\big) := \int_{R^n} ||x-y||^{\alpha-n} f(y)\, dy,$$

$p \geq 1, \quad 1 - \dfrac{1}{p} < \dfrac{\mathrm{Re}\alpha}{n} < 1$ *and $A^\alpha f$ exist at point $x \in R^n$, then the function $A^\alpha f$ is L_p continuous at x.* □

References

[1] E. R. Love and V. K. Tuan, *L^p–Continuity of Reisz Potentials, Integral Transforms and Special Functions*, Vol. 1, No 1, pp 27–31, (1993).

[2] E. M. Stein, *Singular Integrals and Differentiality Properties of Functions*, Princeton University Press, (1970).

[3] A. Zigmund, *Trigonometric series, I–II*, Cambridge, (1968).