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ON THE GENERALIZED LOGISTIC AND LOG-LOGISTIC DISTRIBUTIONS

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Abstract. In this paper, the generalized logistic and the generalized log-logistic distributions are considered. Some theorems that characterize the generalized logistic distribution are proved. Furthermore, a generalization of the log-logistic distribution is defined and its moments are determined. It is pointed out that the t-approximation for the F-distribution proposed in Ojo (1985) can be used to evaluate the cumulative distribution function of the generalized log-logistic distribution. Finally, some relationships between the generalized log-logistic and other distributions are established.

1. INTRODUCTION

The role of the logistic distribution, whose density function is defined as

$$f_X(x) = \frac{e^x}{(1+e^x)^2}, \qquad -\infty < x < \infty \tag{1.1}$$

and its distribution function is given as

$$F_X(x) = \frac{e^x}{(1+e^x)}, \qquad -\infty < x < \infty \tag{1.2}$$

in modeling various stochastic phenomena is well known. Extensive research work has been carried out by several authors investigating the properties and applications of the logistic model (see for example Berkson, (1944); Cox, (1970); Johnson and Kotz, (1970)). A generalization of this distribution whose density function is defined as

$$f_X(x;p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\frac{e^{px}}{(1+e^x)^{p+q}}, \quad -\infty < x < \infty,\ p>0,\ q>0 \tag{1.3}$$

has earlier on been considered by George and Ojo (1980). They obtained the cumulants of the distribution and demonstrated that the cumulative distribution function can be well approximated by the t-distribution. In the present paper, further researches on the generalized logistic and related distribution are carried out. Specifically, some theorems that characterize the generalized logistic distribution are stated and proved. The importance of a related distribution namely the log-logistic distribution was not noticed until recent times. Ali and Khan, (1987); Ali and Umbach, (1990). The random variable X is said to have a log-logistic distribution if lnX is logistic. That is $Pr(lnX \leq x) = (1+e^{-x})^{-1}$.

That is $Pr(X \leq e^x) = (1+e^{-x})^{-1}$.

That is $Pr(X \leq y) = (1+\frac{1}{y})^{-1}$ and so the density function of the log-logistic random variable X is given as

$$f_X(x) = \frac{1}{(1+x)^2}, \quad 0 < x < \infty. \tag{1.4}$$

In this paper a generalization of log-logistic distribution is considered. The moments of the distribution are obtained and some theorems relating this distribution to some statistical distributions are proved.

2. CHARACTERIZATIONS OF THE GENERALIZED LOGISTIC DISTRIBUTION

In this section, some theorems that characterize the generalized logistic distribution are stated and proved.

Theorem 2.1 *Let X be a continuously distributed random variable with density function $f_X(x)$. Then the random variable $Y = ln(e^x - 1)$ is a generalized logistic random variable with parameter $(1,q)$ if and only if X follows an exponential distribution with parameter q.*

Proof. Suppose X has exponential distribution with parameter q,

$$f_X(x;q) = qe^{-qx}, \quad x > 0, \quad q > 0 \tag{2.1}$$

Let $y = ln(e^x - 1)$, by transformation of random variable, we found that

$$f_Y(y) = \frac{qe^y}{(1+e^y)^{1+q}}, \quad -\infty < y < \infty \tag{2.2}$$

which is the probability density function of a generalized logistic random variable Y with parameter $(1, q)$.

Conversely, if Y is a generalized logistic random variable with shape parameter$(1, q)$, then the characteristic function of Y is given as

$$\phi_Y(t) = E[e^{itY}]$$

that is

$$\frac{\Gamma(1+it)\Gamma(q-it)}{\Gamma(q)} = \int (e^x - 1)^{it} f_X(x)dx. \tag{2.3}$$

The only function $f_X(x)$ satisfying equation (2.3) is the $f_X(x)$ given in equation (2.1). This proved the theorem.

Theorem 2.2 *Suppose a continuously distributed random variable X has at-distribution with k degrees of freedom. Then the random variable $Y = ln(x^2/k)$ is distributed according to the generalized logistic distribution with parameters ($p = \frac{1}{2},\ q = \frac{k}{2}$).*

Proof. A random variable X has a t-distribution with k degrees of freedom if

$$f_X(x) = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt(\pi k)}(1+\frac{x^2}{k})^{-(\frac{k+1}{2})}, \quad -\infty < x < \infty. \tag{2.4}$$

Since $y = ln(x^2/k)$, then $x = \pm\sqrt(k)e^{y/2}$. Let $g_1^{-1}(y) = \sqrt(k)e^{y/2}$ and $g_2^{-1}(y) = -\sqrt(k)e^{y/2}$. So, $d[g_i^{-1}(y)]/dy = \pm\frac{1}{2}\sqrt(k)e^{y/2}$.

Therefore

$$f_Y(y) = \sum_{i=1}^{2} |d[g_i^{-1}(y)]/dy| f_X(g_i^{-1}(y)). \tag{2.5}$$

After substitution and simplification, we have

$$f_Y(y) = \frac{\Gamma(\frac{k}{2}+\frac{1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})} \frac{e^{\frac{y}{2}}}{(1+e^y)^{\frac{k}{2}+\frac{1}{2}}}, \quad -\infty < y < \infty \tag{2.6}$$

which is the probability density function for generalized logistic random variables with parameters $(\frac{1}{2}, \ \frac{k}{2})$. Conversely, if Y is a generalized logistic random variable, then the characteristic function of Y is given as

$$\phi_Y(t) = E[e^{ity}] = \int e^{itln(x^2/2)} f_X(x)dx = \int_{-\infty}^{\infty} (\frac{x^2}{k})^{it} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{(\pi k)}} (1+\frac{x^2}{k})^{-(\frac{k+1}{2})} dx. \tag{2.7}$$

By suitable change of variable, we have

$$\phi_Y(t) = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})} \int_0^{\infty} \frac{z^{it-\frac{1}{2}}}{(1+z)^{\frac{k+1}{2}}} dz$$

$$= \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})} B(it+\frac{1}{2}, \frac{k}{2} - it) = \frac{\Gamma(\frac{1}{2}+it)\Gamma(\frac{k}{2}-it)}{\Gamma(\frac{k}{2})\Gamma(\frac{1}{2})} \tag{2.8}$$

which is the characteristic function of a generalized logistic distribution with parameters $(\frac{1}{2}, \frac{k}{2})$. By the uniqueness theorem, the proof is established.

Theorem 2.3 *The random variable X is generalized logistic with distribution functionF given by equation (1.2) if and only if F satisfies the homogeneous differential equation*

$$(e^{2x}-1)F^2F' + e^{2x}F'' = 0 \tag{2.9}$$

for $p = q = 1$ (prime denotes differentiation).

Proof. Suppose X is a generalized logistic random variable with $p = q = 1$, its distribution function is

$$F = \frac{e^x}{(1+e^x)}.$$

It is easily shown that the F above satisfies equation (2.9).

Conversely, if we assume that F satisfies the equation (2.9). Separating the variables in (2.9) and integrating, we have $F = (e^{-x} - k)^{-1}$ where k is a constant. The value of k that makes F a distribution function is $k = -1$.

Possible Application of Theorem 2.3

From equation (2.9), we have

$$x = \frac{1}{2} ln(\frac{F^2 F'}{F^2 F' + F''}) \tag{2.10}$$

where F is the distribution function as written above. Thus, the importance of theorem (2.3) lies in the linearising transformation (2.10). The transformation (2.10) can be regarded as another alternative to Berkson's logit transform (Berkson, 1944) and Ojo's logit transform when $p = q = 1$ (Ojo, 1997) for the ordinary logistic model.

Thus in the analysis of bioassay and quantal response data, if model (1.1) is used, what Berkson's logit transform does for the ordinary logistic can be done for the model (1.1) by the transformation (2.10).

3. ON GENERALIZED LOG-LOGISTIC DISTRIBUTION

3.1 The generalized log-logistic distribution and its moments

The random variable Y is said to have the generalized log-logistic distribution if lnY is generalized logistic. That is

$$Pr(lnY \leq y) = \frac{1}{B(p,q)} \int_{-\infty}^{y} \frac{e^{px}}{(1+e^x)^{p+q}} dx.$$

That is

$$Pr(Y \leq e^y) = \frac{1}{B(p,q)} \int_{-\infty}^{y} \frac{e^{px}}{(1+e^x)^{p+q}} dx.$$

That is

$$Pr(Y \leq u) = \frac{1}{B(p,q)} \int_{-\infty}^{lnu} \frac{e^{px}}{(1+e^x)^{p+q}} dx.$$

Thus, the density function of Y by differentiation under the integral is given as

$$g(y) = \frac{1}{B(p,q)} \frac{y^{p-1}}{(1+y)^{p+q}}, \quad 0 < y < \infty. \tag{3.3}$$

This generalized version of the log-logistic distribution has ealier on been used to analyse some survival data in Mohammed et al (1992). The characteristic function of Y is given as

$$\phi_Y(t) = \frac{1}{B(p,q)} \int_0^{\infty} \frac{y^{p-1} e^{ity}}{(1+y)^{p+q}} dy = \sum_{k=0}^{\infty} \frac{i^k t^k}{k!} \frac{\Gamma(p+k)\Gamma(q-k)}{\Gamma(p)\Gamma(q)}. \tag{3.4}$$

By direct integration, the r^{th} moment of Y is given as

$$\mu_r = \frac{1}{B(p,q)}\int_0^\infty \frac{y^{p+r-1}}{(1+y)^{p+q}}dy = \frac{\Gamma(p+r)\Gamma(q-r)}{\Gamma(p)\Gamma(q)}$$

$$= \frac{p(p+1)(p+2)...(p+r-1)}{(q-1)(q-2)(q-3)...(q-r)}. \tag{3.5}$$

In particular the first four central moments are given as

$$\mu_1 = \frac{p}{q-1},$$

$$\mu_2 = \frac{p(p+1)}{(q-1)(q-2)},$$

$$\mu_3 = \frac{p(p+1)(p+2)}{(q-1)(q-2)(q-3)},$$

$$\mu_4 = \frac{p(p+1)(p+2)(p+3)}{(q-1)(q-2)(q-3)(q-4)}.$$

3.2 The cumulative distribution function of Y

The cumulative distribution function of Y can be evaluated by using the t-approximation proposed in Ojo (1985) for the F-distribution (see theorem 4.2 of this paper). For the purpose of this paper, we recall the t-approximation for the F-distribution as contained in Ojo (1985). This approximation was given as

$$P[F(2q,2p) \geq \frac{p}{q}e^{-(\kappa_1+\frac{t_\nu}{c})}] \sim P[T \leq t_\nu]$$

or equivalently interms of percentile

$$F_o(2q,2p) \sim \frac{p}{q}e^{-(\kappa_1+\frac{t_\nu}{c})}$$

where F_o denotes the upper percentage point of the F distribution and t_ν denotes the lower percentile of the t distribution with ν degrees of freedom.$c = \frac{\sigma_t}{\sqrt{(\kappa_2)}}$, σ_t being the standard deviation of t distribution and κ_i the i^{th} cumulant of the generalized logistic distribution. Consequently, the approximation for the distribution function of the generalized log-logistic can easily be obtained.

3.3 Some theorems relating the generalized log-logistic to other distributions

Theorem 3.1 *Let X be a continuously distributed random variable. Then the random variable $Y = \frac{X}{1-X}$ has the generalized log-logistic distribution if X has beta distribution with parameters (p, q).*

Proof. If X is beta (p, q), then

$$Pr(Y \leq y) = Pr[(\frac{X}{1-X}) \leq y]$$

$$= Pr(X \leq \frac{y}{1+y}) = \int_0^{\frac{y}{1+y}} \frac{1}{B(p,q)} t^{p-1}(1-t)^{q-1} dt.$$

By differentiating under the integral, the density function of $Y = \frac{X}{1-X}$ is given as

$$g(y) = \frac{1}{B(p,q)} (\frac{y}{1+y})^{p-1} (1 - \frac{y}{1+y})^{q-1} \cdot (\frac{1}{(1+y)^2})$$

$$= \frac{1}{B(p,q)} \frac{y^{p-1}}{(1+y)^{p+q}}, \quad 0 < y < \infty.$$

This completes the proof.

Theorem 3.2 *Let $F(2q, 2p)$ be an F-random variable which has an F-distribution with $(2q, 2p)$ degrees of freedom. The random variable $Y = \frac{q}{p}F(2q, 2p)$ has a generalized log-logistic distribution with parameter (p, q).*

Proof. The density function of an $F(2q, 2p)$ random variable can be written as

$$f_X(x) = K \frac{x^{q-1}(\frac{q}{p})^{q-1}}{(1 + \frac{q}{p}x)^{p+q}}, 0 < x < \infty \tag{3.6}$$

where K is the normalizing constant. By omitting all constant, the density of Y can be written as

$$g(y) \propto \frac{y^{q-1}}{(1+y)^{p+q}} \tag{3.7}$$

Since any density function proportional to the right hand side of (3.7) is that of a generalized log-logistic random variable, the proof is complete.

Theorem 3.3 *Let $X_1,\ X_2$ be gamma random variables with density functions*

$$h_1(x_1) = \frac{1}{\Gamma(p)} x_1^{p-1} e^{-x_1}, \quad x_1 \geq 0$$

$$h_2(x_2) = \frac{1}{\Gamma(q)} x_2^{q-1} e^{-x_2}, \quad x_2 \geq 0$$

Then, the random variable $Y = \frac{X_1}{X_2}$ has the generalized log-logistic distribution if X_1 and X_2 are independent.

Proof. The joint density function of X_1 and X_2 is

$$f(x_1, x_2) = \frac{1}{\Gamma(p)} \frac{1}{\Gamma(q)} x_1^{p-1} x_2^{q-1} e^{-(x_1+x_2)}. \tag{3.8}$$

Let $y_1 = \frac{x_1}{x_2}$ and $y_2 = x_2$ so the density of Y_1 is

$$\begin{aligned} g(y_1) &= \frac{1}{\Gamma(p)} \frac{1}{\Gamma(q)} \int_0^\infty (y_1 y_2)^{p-1} y_2^{q-1} y_2 e^{-y_2(1+y_1)} dy_2 \\ &= \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \frac{y_1^{p-1}}{(1+y_1)^{p+q}}. \end{aligned} \tag{3.9}$$

Thus the density of $Y = \frac{X_1}{X_2}$ is

$$g(y) = \frac{1}{B(p,q)} \frac{y^{p-1}}{(1+y)^{p+q}}\ , \quad 0 < y < \infty$$

This proves the theorem.

Theorem 3.4 *Let X be a generalized log-logistic random variable with parameters (p, q). The random variable $Y = \sqrt{(\nu X)}$ has a t distribution with ν degree of freedom if $p = \frac{1}{2}$ and $q = \frac{\nu}{2}$.*

Proof. Let $g(y)$ denote the density function of Y and let $y = \sqrt{(\nu x)}$, the density function of the generalized log-logistic is

$$f_X(x) = \frac{1}{B(p,q)} \frac{x^{p-1}}{(1+x)^{p+q}}.$$

Then

$$g(y) \propto \frac{y^{2p-2}}{(1+\frac{y^2}{\nu})^{p+q}} y = \frac{y^{2p-1}}{(1+\frac{y^2}{\nu})^{p+q}} \tag{3.10}$$

If $p = \frac{1}{2}$ and $q = \frac{\nu}{2}$, then

$$g(y) \propto (1 + \frac{y^2}{\nu})^{-\frac{1}{2}(1+\nu)}$$

Since any random variable whose density is proportional to $(1 + \frac{y^2}{\nu})^{-\frac{1}{2}(1+\nu)}$ has a t-distribution with ν degree of freedom, the theorem follows.

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