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ASYMPTOTIC BEHAVIOR FOR THE BEST LOWER BOUND OF JENSEN'S FUNCTIONAL

Stojan Radenović[1] and Mirjana Pavlović[2]

[1] *University of Belgrade, Faculty of Mechanical Engineering, 27. marta 80, 11000 Belgrade, Serbia and Montenegro*
[2] *University of Kragujevac, Faculty of Science, Department of Mathematics, P.O.Box 60, 34000 Kragujevac, Serbia and Montenegro*

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Abstract. In this note we consider asymptotic estimates for the best lower bound of Jensen's functional $f \mapsto \int_0^{2\pi} \log|\, f(e^{i\theta})\,|\frac{d\theta}{2\pi}$, f satisfies (1) and (2) below, when $k \to +\infty$.

Let $f(z) = \sum_{j=0}^{n} a_j z^j$ $(\neq 0)$ be a polynomial with complex coefficients and let d be a real number such that $0 < d < 1$. Then $f(z)$ is said to have concentration d at degrees at most k, measured by the l_p-norm, $(1 \leq p \leq 2)$, if

$$\left(\sum_{j \leq k} |a_j|^p\right)^{\frac{1}{p}} \geq d \cdot \left(\sum_{j \geq 0} |a_j|^p\right)^{\frac{1}{p}} . \tag{1}$$

Polynomials with concentrations of low degrees introduced by B.Beauzamy and P.Enflo, who proved, for such polynomials, a generalized Jensen's inequality [1] and [2]; this plays an important role in the construction of an operator on a Banach space with no non-trivial invariant subspace [4].

We investigate here the estimates of Jensen's functional $f \mapsto \int\limits_0^{2\pi} \log\left(\frac{|f(e^{i\theta})|}{|f|_{l_p}}\right)\frac{d\theta}{2\pi}$ for polynomials satisfying (1).In the sequel, we shall normalize f and assume that

$$\left(\sum_{j\geq 0}|a_j|^p\right)^{\frac{1}{p}} = 1\ . \tag{2}$$

For such polynomials, it is shown in [6] the following:

If $f(z) = \sum\limits_{j=0}^{n} a_j z^j$ is a polynomial which satisfies (1) and (2), then:

$$\int\limits_0^{2\pi} \log|\ f(e^{i\theta})\ |\frac{d\theta}{2\pi} \geq C(d,k,p) = \max_{1<t<+\infty} f_{d,k,p}(t)\ , \tag{3}$$

where

$$f_{d,k,p}(t) = \begin{cases} \frac{t}{p}\cdot\log\frac{d^p[(\frac{t+1}{t-1})^p-1]}{(\frac{t+1}{t-1})^{p(k+1)}-1} - \frac{1}{2}t^2, & 1<p\leq 2 \\ t\cdot\log\frac{2d}{(t-1)[(\frac{t+1}{t-1})^{k+1}-1]}, & p=1 \end{cases}.$$

For our purpose here, $C(d,k,p)$ will denote the largest such constant possible in (3), i.e.

$$C(d,k,p) := \inf\left\{\int\limits_0^{2\pi}\log|\ f(e^{i\theta})\ |\frac{d\theta}{2\pi}\ :\ f \text{ satisfies (1) and (2)}\right\}. \tag{4}$$

In the sequel, we have restricted ourselves to the numerical asymptotic estimates for $C(d,k,p)$, when $k \to +\infty$, $0 < d < 1$ and $1 < p \leq 2$.

Theorem. *Let $p \in\ ]1,2]$ and $d \in\ ]0, 2^{-\frac{1}{p}}]$. Then for sufficiently large k the best constant $C(d,k,p)$ satisfies : $-2k \leq C(d,k,p) \leq -2k\cdot\log 2$.*

Proof. Firstly, we represent $f_{d,k,p}(t)$ in the form:

$$f_{d,k,p}(t) = h_{d,p}(t) + g_k(t) - \frac{t}{p}\cdot\log\left[1-\left(\frac{t-1}{t+1}\right)^{p(k+1)}\right],$$

where

$$\begin{aligned} h_{d,p}(t) &= t\cdot\log d - \frac{1}{2}t^2 + \frac{t}{p}\cdot\log[(t+1)^p-(t-1)^p] \\ g_k(t) &= kt\cdot\log(t-1)-(k+1)t\cdot\log(t+1). \end{aligned}$$

It is clear that $f_{d,k,p}(t) > h_{d,p}(t) + g_k(t), t > 1$. We shall now prove that the function $h_{d,p}(t) + g_k(t)$ takes its maximum value at a point (unique) t_k such that

$t_k \to +\infty$, when $k \to +\infty$. We shall maximize $h_{d,p}(t) + g_k(t)$, since the remaining term $t \cdot \log \left[1 - \left(\frac{t-1}{t+1}\right)^{p(k+1)}\right]$ can be neglected, for $t = t_k$, when $k \to +\infty$. We now find derivatives for $h_{d,p}(t)$ and $g_k(t)$.

$$\begin{aligned}
g'_k(t) &= -(k+1)\cdot \log(t+1) + k\cdot \log(t-1) - \frac{(k+1)t}{t+1} + \frac{kt}{t-1};\\
g''_k(t) &= -\frac{(t-1)^2(t+2)+4k}{(t^2-1)^2} < 0;\\
h'_{d,p}(t) &= \log d - t + \frac{1}{p}\cdot \log[(t+1)^p - (t-1)^p] + t\cdot \frac{(t+1)^{p-1}-(t-1)^{p-1}}{(t+1)^p-(t-1)^p};\\
h''_{d,p}(t) &= -1 + 2\cdot \frac{(t+1)^{p-1}-(t-1)^{p-1}}{(t+1)^p-(t-1)^p}\\
&\quad + \frac{t(p-1)}{A^2(t)}\left(\left[(t+1)^{p-2}-(t-1)^{p-2}\right]A(t) - p\left[(t+1)^{p-1}-(t-1)^{p-1}\right]^2\right),
\end{aligned}$$

where $A(t) = (t+1)^p - (t-1)^p$.

Since $p \in\ ]1,2]$, $t > 1$, it is clear that

$$h''_{d,p} < 0 \text{ iff } -1 + 2\cdot \frac{(t+1)^{p-1}-(t-1)^{p-1}}{A(t)} < 0.$$

But, this is true iff $\varphi_p(t) < 0$, where $\varphi_p(t) = 2(t+1)^{p-1} - 2(t-1)^{p-1} - (t+1)^p + (t-1)^p$. Hence, we find that

$$\varphi'_p(t) = 2(p-1)\left[\left((t+1)^{p-2} - (t-1)^{p-2}\right)\right] + p\left[(t-1)^{p-1} - (t+1)^{p-1}\right] < 0.$$

This shows that $h''_{d,p}(t) + g''_k(t) < 0$. Since

$$\lim_{t\to 1+}(h'_{d,p}(t) + g'_k(t)) = +\infty \text{ and } \lim_{t\to+\infty}(h'_{d,p}(t) + g'_k(t)) = -\infty,$$

equation $h'_{d,p}(t) + g'_k(t) = 0$ has exactly one solution t_k. From the equality $h'_{d,p}(t) + g'_k(t) = 0$ we get with $t = t_k$,

$$k = \frac{(t^2-1)\cdot \log(t+1) + t^2 - t - (t^2-1)h'_{d,p}(t)}{2t + (t^2-1)\cdot \log(t-1) - (t^2-1)\cdot \log(t+1)}, \tag{5}$$

wherefrom we easily deduce that $t_k \to +\infty$ iff $k \to +\infty$.

Writing $\log(t \pm 1) = \log t + \log(1 \pm \frac{1}{t})$, $(t \pm 1)^p = t^p(1 \pm \frac{1}{t})^p$ and substituting Taylor expansion of order 3 when $k \to +\infty$, we get

$$k \sim \frac{3}{4}t_k^4. \tag{6}$$

So we have shown that at the point t_k, the value of $f_{d,k,p}(t)$ and the value of $h_{d,p}(t) + g_k(t)$ are asymptotically the same. All we have to do now is to compute $f_{d,k,p}(t_k)$, and this follows easily by the estimate (6) and the asymptotic equalities for $\log(1 \pm \frac{1}{t})$ and $(1 \pm \frac{1}{t})^p$:

$$\begin{aligned} f_{d,k,p}(t_k) &\sim h_{d,p}(t_k) + g_k(t_k) = \left(-\frac{1}{2}t_k^2\right)(1+o(1)) + (-2k)(1+o(1)) \\ &= (-2k)\left(1 + o(1) + \frac{t_k^2}{4k}\right) = (-2k)(1+o(1)) \sim -2k, \end{aligned}$$

because $\frac{t_k^2}{4k} \sim \frac{t_k^2}{4\cdot\frac{3}{4}t_k^4} = \frac{1}{3t_k^2} \to 0,\ k \to +\infty$. This proves the asymptotic estimate $C(d,k,p) \geq f_{d,k,p}(t_k) \sim -2k,\ k \to +\infty$, for each $d \in\ ]0,1[$ and for each $p \in\ ]1,2]$. If $p = 1$, from [2] it follows that $k \sim \frac{3}{4}t_k^3 \log t_k,\ k \to +\infty$.

Consider now the polynomial $g(z) = \frac{1}{|g_1|_{l_p}} \cdot g_1(z)$ where $g_1(z) = (\frac{1}{2} + \frac{z}{2})^{2k+1}$. It satisfies (1) and (2). By the properties of the binomial coefficients, this polynomial has concentration $d \leq 2^{-\frac{1}{p}}$ at degrees k, measured by the l_p-norm. Indeed, from

$$\sum_{j\leq k} \frac{1}{|\ g_1\ |_{l_p}^p \cdot 2^{(2k+1)p}} \binom{2k+1}{j}^p \geq d^p \sum_{j\geq 0} \frac{1}{|\ g_1\ |_{l_p}^p \cdot 2^{(2k+1)p}} \binom{2k+1}{j}^p,$$

it follows

$$\sum_{j\leq k} \frac{1}{2^{(2k+1)p}} \binom{2k+1}{j}^p \geq d^p 2 \cdot \sum_{j\leq k} \frac{1}{2^{(2k+1)p}} \binom{2k+1}{j}^p,$$

that is $0 < d \leq 2^{-\frac{1}{p}}$. But the constant term is $\frac{1}{|g_1|_{l_p}\cdot 2^{(2k+1)}}$, the only root is -1, so Jensen's formula says that:

$$\begin{aligned} \int_0^{2\pi} \log |\ g(e^{i\theta})\ | \frac{d\theta}{2\pi} &= -(2k+1)\cdot \log 2 - \log |\ g_1\ |_{l_p} \\ &= (-2k \log 2)(1+o(1)) \sim -2k\log 2. \end{aligned}$$

Hence, we have asymptotically: $C(d,k,p) \leq -2k \cdot \log 2$, when $k \to +\infty$, for each $d \in\ ]0, 2^{-\frac{1}{p}}]$ and for each $p \in\ ]1,2]$. The same estimate is true and in case $p = 1$ (see[2]).

Remark. If $k = 0$, it follows that $f_{d,0,p}(t) = t\log d - \frac{1}{2}t^2, 1 < p \leq 2$, as and $f_{d,0,1}(t) = t \log d$. Now, we obtain that $C(d,0,p) = \log d - \frac{1}{2}, 1 < p \leq 2$, i.e.

$C(d,0,1) = \log d$. Hence, there exists the precise value of the best constant $C(d,0,p)$. This generalizes Theorem 1 from [2]. Our Theorem also generalizes the corresponding result in ([1], Lemme 3) for $p = 2$. It should be noted that in case $k > 0$, nothing is known about the precise value of $C(d,k,p)$, even for small values of k.

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