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RELATIONS BETWEEN INTRINSIC AND EXTRINSIC CURVATURES

S. Haesen[1], A. Šebeković[2] and L. Verstraelen[3]

[1]*K.U.Brussel, Vrijheidslaan 17, B-1081 Brussel, Belgium;*
[2]*University of Kragujevac, Department of Mathematics, Radoja Domanovica 12, 34000 Kragujevac, Serbia and Montenegro;*
[3]*K.U.Leuven, Celestijnenlaan 200B, B-3001 Heverlee (Leuven), Belgium.*

Introduction

In some sense reversing the historical traject, in §1 it will be indicated how all *scalar-valued intrinsic curvatures of Riemannian manifolds* can be determined in terms of the *curvatures of associated Euclidean curves.* This involves the consideration of *arbitrary-dimensional normal sections of submanifolds in Euclidean spaces* and their *projections* on appropriate subspaces. In terms of such normal sections of Euclidean submanifolds and of such projections, in §2 some comments will be made concerning *general inequalities* for Euclidean submanifolds between their *scalar curvature* and their *mean-* and *normal scalar curvatures.*

1. Extrinsic views on intrinsic curvatures

In accordance with our intuition, *the curvature of curves in Euclidean planes* was determined around 1670 by Newton, using the notion of *osculating circles*, (cf. [1]). And, in terms of the curvature of Euclidean planar curves, using Euler's notion of *normal sections of surfaces in M^2*

in Euclidean 3-dimensional spaces $\mathbb{E}^3$, since 1760, also the curvature behaviour of such surfaces became well describable. Depending on the nature of the problems under investigation, as combinations of the *principal curvatures*, various *geometrical curvature characteristics of surfaces* M^2 *in* $\mathbb{E}^3$ were introduced and studied, such as the mean curvature of Germain, the Gauss curvature, the Casorati curvature (or "curvedness") and the shape-index of Koenderink-van Doorn (cf. [2] [3]).

The *theorema* which asserts the invariance of the Gauss curvature under isometric deformations of surfaces M^2 in $\mathbb{E}^3$ is indeed *egregium*, as Gauss labeled it in his general theory of curved surfaces and as its impact on the development of mathematics has amply shown, (cf. [4]). Amongst others, it immediately yielded the distinction between the *intrinsic* and the *extrinsic* qualities of such surfaces. And, as Gauss anticipated, it led to the creation in 1854 of, amongst others, the *Riemannian geometry*, which, in the words of Chern [5], forms the core of modern differential geometry. The main characteristics of Riemannian spaces are their curvatures, (cf. [6]). Essentially, all curvature information regarding a Riemannian manifold (M^n, g) is contained in its *Riemann-Christoffel curvature tensor* R, or, equivalently, in the knowledge of its *Riemannian* or *sectional curvatures.* As shown by Riemann, the sectional curvature $K(P)$ of (M^n, g) for the plane P spanned by any two orthonormal tangent vectors X and Y at any of its points $p, K(P) = R(X, Y; Y, X)$, is the intrinsically determinable Gauss curvature $K_{G^2_P}(p)$, i.e. : the Gauss curvature at p of the 2-dimensional surface G^2_P which is formed by the *geodesics* of (M^n, g) passing through p and tangent to P.

By the *isometric inbedding theorem* of Nash, all n-dimensional Riemannian manifolds (M^n, g) can be seen as submanifolds M^n of Euclidean spaces $\mathbb{E}^{n+m}$ with certain codimensions m, (for basic notions and results on Riemannian submanifolds, see [7] [8]). Let M^n be *a submanifold of a Euclidean space* $\mathbb{E}^{n+m}$ and let P be any 2-dimensional section of its tangent space T_pM^n at any of its points p. Let $(e_1, \cdots, e_n, \xi_1, \cdots, \xi_m)$ be any adapted orthonormal frame around p on M^n in $\mathbb{E}^{n+m}$, whereby $e_1, \cdots, e_n$ and $\xi_1, \cdots, \xi_m$ are respectively tangent and normal to M^n and such that, at p, $e_i \wedge e_j = P$, $(i, j \in \{1, \cdots, n\}; \alpha, \beta \in \{1, \cdots, m\})$. Then, the equation of Gauss for M^n in $\mathbb{E}^{n+m}$ shows that the intrinsic Riemannian curvature $K(P)$ of (M^n, g) is given by $K(P) = K_{ij} = \sum_\alpha [h^\alpha_{ii} h^\alpha_{jj} - (h^\alpha_{ij})^2](p)$, whereby h denotes the second fundamental form of M^n in $\mathbb{E}^{n+m}$. Now, let $\sum^2_P$ be *the 2-dimensional* *normal section of the submanifold M^n of $\mathbb{E}^{n+m}$ corresponding to P*, i.e. : the intersection of M^n with the affine $(2+m)$-dimensional subspace of $\mathbb{E}^{n+m}$ spanned by P and $T_p^{\perp}M^n$, the normal space of M^n at p. This normal section $\sum_P^2$, further also denoted by M_{ij}^2, thus is a surface in $\mathbb{E}^{2+m}$. Its equation of Gauss shows that its Gauss curvature $K_{\sum_P^2}(p)$ at p is given by $K_{\sum_P^2}(p) = \sum_\alpha [h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2](p)$.

Theorem 1. *The sectional curvature $K(P)$ of any submanifold M^n of a Euclidean space $\mathbb{E}^{n+m}$, for any tangent plane section P at any of its points p, equals the Gauss curvature at p of the corresponding 2-dimensional normal section $\sum_P^2$ of M^n in $\mathbb{E}^{n+m}$.*

The summants $[h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2](p)$ occuring above, (called "partial curvatures" by Ricci), can readily be seen to equal the Gauss curvatures at p of the surfaces $M_{ij\alpha}^2$ in $\mathbb{E}^3$ which are the *projections* of the normal sections $M_{ij}^2 = \sum_P^2$ on the affine 3-dimensional spaces in $\mathbb{E}^{2+m} = P \oplus T_p^{\perp}M^n$ at p which are spanned by $P = e_i \wedge e_j$ and the vectors ξ_α at p (for each individual α). Now, observe that by Euler's formula which expresses all normal curvatures, i.e. the curvatures of all (1-dimensional) normal sections of a surface M^2 in $\mathbb{E}^3$, in terms of its principal curvatures, the Gauss curvature K of any such surface equals $\pm$ the square of the curvature of (in general precisely two such) normal sections, or still, K equals $\pm\pi$ times the inverse of the area of their osculating circles. Therefore, by eventually making an appropriate choice of normal frame $(\xi_1, \cdots, \xi_m)$ on M^n in $\mathbb{E}^{n+m}$, one obtains the following.

Theorem 2. *Any sectional curvature of a submanifold M^n of $\mathbb{E}^{n+m}$ is determined by the curvatures of at most two associated Euclidean planar curves.*

Similarly, eventually making use of *k-dimensional normal sections of submanifolds M^n in $\mathbb{E}^{n+m}$* for appropriate $k \in \{1, \cdots, n-1\}$, also the Riemannian scalar curvature and the scalar-valued curvature invariants introduced since 1993 by B.Y. Chen (as his so-called δ-curvatures and their very recently introduced refinements for specific classes of semi-Riemannian manifolds, cf. [8] [9] [10] [11]), can be determined by the curvatures of associated curves.

2. Inequalities between intrinsic and extrinsic curvatures

The series of *optimal inequalities* between scalar-valued *intrinsic-Chen curvatures* of a submanifold M^n in $\mathbb{E}^{n+m}$ and its *extrinsic squared mean curvature* H^2, and several related studies that originated in this context, amongst others, can be considered as first systematic steps in making an effective use and in achieving a better understanding of the isometric inbedding theorem of Nash, (cf. [8] [9] [10] [11]; similar inequalities were, of course, also established for submanifolds of the "nicest" non-Euclidean semi-Riemannian ambient spaces). Concerning similar inequalities involving in addition *extrinsic scalar-valued normal curvatures*, as far as we know, the only general result is the inequality $\rho \leq H^2 - \rho^\perp$, $(*)$, whereby ρ and $\rho^\perp$ respectively denote the *normalised scalar curvature* and the *normalised normal scalar curvature* of M^n in $\mathbb{E}^{n+m}$ (and in the other real space forms), which is proven to be valid for $n = 2$ and arbitrary m and for arbitrary n and $m = 2$, (cf. [12][13][14][15]; in case $n = 2$ *the normal curvature* $\rho^\perp$ is *the area of the curvature ellipse* modulo 2π, and $(*)$was conjectured in [15] to hold for all dimensions n and codimension m). In the cases of either 2-dimensional or 2-codimensional submanifolds, $(*)$ is an equality if and only if the second fundamental form h has a very specific expression (asserting geometrically for $n = 2$ that the curvature ellipse then actually is a circle) and many concrete examples are known of submanifolds realising the equality in $(*)$.

As an illustration of possible applications of arbitrary dimensional normal sections of submanifolds and of their projections mentioned above, their use will be indicated now in obtaining such new optimal inequalities which involve other scalar-valued normal curvatures of Euclidean submanifolds. In the following explicitations, hereby restriction will be made to 2-dimensional normal sections. Let M^n be a submanifold in $\mathbb{E}^{n+m}$ with an adapted frame $(e_1, \cdots, e_n, \xi_1, \cdots, \xi_m)$. For any tangent plane section $P = e_i \wedge e_j$, for any $i \neq j$, consider, for all α, the *projections* $M^2_{ij\alpha}$ in $\mathbb{E}^3$ of the corresponding *normal section* $\sum^2_P = M^2_{ij}$ in $\mathbb{E}^{2+m}$. Let k^α_1 and k^α_2 be the *principal curvatures* of the surface $M^2_{ij\alpha}$ in $\mathbb{E}^3$. Then its *Gauss* and *mean curvature* are respectively given by $K(M^2_{ij\alpha}) = k^\alpha_1 k^\alpha_2$ and $H(M^2_{ij\alpha}) = \frac{1}{2}(k^\alpha_1 + k^\alpha_2)$, and $K(M^2_{ij\alpha}) \leq H^2(M^2_{ij\alpha})$ whereby equality holds if and only if $k^\alpha_1 = k^\alpha_2$. Thus, for all tangent indices $i \neq j$ and normal indices α one has $4[h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2] \leq (h^\alpha_{ii} + h^\alpha_{jj})^2$ whereby equality holds if and only if $h^\alpha_{ij} = 0$ and $h^\alpha_{ii} = h^\alpha_{jj}$. Summing up these inequalities over all α yields

$4K(P) = 4K_{ij}$ on the left hand side, and next summing up over all $i \neq j$ by taking into account that the *Riemannian scalar curvature* τ of M^n and the squared *mean curvature* H^2 of the submanifolds M^n in $\mathbb{E}^{n+m}$ are respectively given by $\tau = \sum_{i<j} K_{ij}$ and $H^2 = \frac{1}{n^2}\sum_\alpha(\sum_i h^\alpha_{ii})^2$ gives $2n\tau + 2(n-2)\sum_{\alpha,i<j}(h^\alpha_{ij})^2 \leq n^2(n-1)H^2$, whereby equality holds if and only if $h^\alpha_{ij} = 0$ and $h^\alpha_{ii} = h^\alpha_{jj}$ for all α and for all $i < j$. The *normal connection* of M^n in $\mathbb{E}^{n+m}$ is *flat* if and only if all shape operators are simultaneously diagonalisable, i.e. if, for an appropriate frame $h^\alpha_{ij} = 0$ for all $\alpha, i < j$. So the quantity $\sum_{\alpha,i<j}(h^\alpha_{ij})^2 \geq 0$ is a certain scalar measure for the deviation from triviality of the normal bundle of M^n in $\mathbb{E}^{n+m}$. For simplicity of formulation, we put $\rho = \frac{2}{n(n-1)}\tau$ (the normalised scalar curvature of the Riemannian manifold M^n) and $\kappa^\perp_1 = \inf \frac{2(n-2)}{n^2(n-1)}\sum_{\alpha,i<j}(h^\alpha_{ij})^2$, whereby inf is taken over all adapted frames on M^n in $\mathbb{E}^{n+m}$. Then, the above general pointwise inequality becomes : $\rho \leq H^2 - \kappa^\perp_1$ whereby equality holds everywhere if and only if M^n is (a part of) an n-plane or a round n-sphere in $\mathbb{E}^{n+m}$. Similarly, considering the *projections* $M^2_{ij\alpha\beta}$ on the space $\mathbb{E}^4$ given by the affine 4-plane $e_i \wedge e_j \wedge e_\alpha \wedge e_\beta (i \neq j, \alpha \neq \beta)$ of the *2-dimensional normal sections* M^2_{ij} in $\mathbb{E}^{2+m}$ of M^n in $\mathbb{E}^{n+m}$, the *Wintgen-inequality* for $M^2_{ij\alpha\beta}$ in $\mathbb{E}^4$ states that $[h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2] + [h^\beta_{ii}h^\beta_{jj} - (h^\beta_{ij})^2] \leq \frac{1}{4}[(h^\alpha_{ii} + h^\alpha_{jj})^2 + (h^\beta_{ii} + h^\beta_{jj})^2] - \frac{1}{2\pi} \cdot A(E_{ij\alpha\beta})$ whereby $A(E_{ij\alpha\beta})$ denotes the *area* of the *curvature ellipse* $E_{ij\alpha\beta}$ of the surface $M^2_{ij\alpha\beta}$ in $\mathbb{E}^4$, and whereby equality holds if and only if $E_{ij\alpha\beta}$ is a circle. Summing up these inequalities over all $i \neq j$ and $\alpha \neq \beta$ then yields the following general pointwise inequality for M^n in $\mathbb{E}^{n+m}$: $\rho \leq H^2 - \kappa^\perp_2$, whereby $\kappa^\perp_2 = \inf[\frac{2(n-2)}{n^2(n-1)}\sum_{\alpha,i<j}(h^\alpha_{ij})^2 + 2\pi \cdot \sum_{\alpha<\beta,i<j} A(E_{ij\alpha\beta})] \geq 0$ and whereby equality is characterised by a specific expression of the second fundamental form h (cf. [13]). Finally, the *Rouxel-Guadalupe-Rodriguez-inequality* for each *2-dimensional normal section* M^2_{ij} in $\mathbb{E}^{2+m}$ itself states that $\sum_\alpha[h^\alpha_{ii}h^\alpha_{jj} - (h^\alpha_{ij})^2] \leq \frac{1}{4}\sum_\alpha(h^\alpha_{ii} + h^\alpha_{jj})^2 - \frac{1}{2\pi} \cdot A(E_{ij})$, whereby $A(E_{ij})$ denotes the *area* of the *curvature ellipse* E_{ij} of the surface M^2_{ij} in $\mathbb{E}^{2+m}$, and whereby equality holds if and only if E_{ij} is a circle. Summing up these inequalities over all $i \neq j$ and $\alpha \neq \beta$ then yields the following general pointwise inequality for M^n in $\mathbb{E}^{n+m}$: $\rho \leq H^2 - \kappa^\perp_m$ whereby $\kappa^\perp_m = \inf[\frac{2(n-2)}{n^2(n-1)}\sum_{\alpha,i<j}(h^\alpha_{ij})^2 + \frac{1}{2\pi} \cdot \sum_{i<j} A(E_{ij})] \geq 0$ and whereby equality holds if and only if the second fundamental form h has a specific expression, (cf. [13][14]). Summarising, thus making use of the 2-dimensional normal sections of submanifolds M^n in $\mathbb{E}^{n+m}$ and of

their projections on Euclidean subspaces with codimension $1, 2, \cdots, m$, respectively, the following results are obtained.

Theorem 3. *Let M^n be any submanifold in a Euclidean space $\mathbb{E}^{n+m}$. Then its intrinsic Riemannian invariant ρ and its extrinsic submanifold invariants H^2 and $\kappa_1^\perp, \kappa_2^\perp, \cdots, \kappa_m^\perp$ (each of whose vanishing charactering that M^n has a trivial normal bundle in $\mathbb{E}^{n+m}$) satisfy the following pointwise general inequalities : $\rho \leq H^2 - \kappa_\alpha^\perp$, whereby equality holds if and only if the second fundamental form h of M^n in $\mathbb{E}^{n+m}$ has a very specific known expression.*

Making use of *higher-dimensional normal sections* of submanifolds M^n in $\mathbb{E}^{n+m}$ and of their *projections*, similarly further such inequalities involving other scalar-valued normal curvatures $\kappa^\perp$ can be obtained and the submanifolds realising the corresponding equalities can be characterised. Along these lines, amongst others, also *new extrinsic normal curvatures* could be introduced, and studied in particular in the context of such inequalities for Euclidean submanifolds, in analogy with the *intrinsic Chen curvatures* of semi-Riemannian manifolds and their recent refinements to special classes of such manifolds.

References

[1] P. Dombrowski : *Wege in euklidischen Ebenen, Kinematik der Speziellen Relativitätstheorie*, Springer, Berlin, 1999.

[2] J.J. Koenderink, *Solid Shape*, The MIT-Press, Cambridge MA, 1990.

[3] W. Kühnel, *Differential geometry : curves, surfaces, manifolds*, AMS, Student mathematical library, 2002.

[4] P. Dombrowski : *150 years after Gauss "disquisitiones generalis circa superficies curvas"*, Astérisque 62, p. 1-153, 1979.

[5] S.S. Chern : *Differential geometry : its past and its future*, Actes 1/CM, Nice, p. 41-53, 1970.

[6] M. Berger : *Riemannian geometry during the second half of the 20-th century*, DMV, 1998.

[7] B.Y. Chen : *Geometry of submanifolds*, Marcel Dekker, New York, 1973.

[8] B.Y. Chen : *Riemannian submanifolds* in *Handbook of Differential Geometry*, Vol. 1, Elsevier, Amsterdam, 2000.

[9] B.Y. Chen : *What can we do with Nash's embedding theorem ?*, preprint.

[10] L. Verstraelen : *Some comments on the δ-curvature of Bang-Yen Chen*, Rend. Sem. Matem. Messina, 1998 International Congres in honor of P. Calapso, p. 331-337.

[11] S. Haesen and L. Verstraelen : *Ideally embedded space-times*, preprint.

[12] P. Wintgen : *Sur l'inégalité de Chen-Willmore*, C.R. Acad. Sc. Paris 288, p. 993-995, 1979.

[13] B. Rouxel : *Sur quelques propriétés des sous-variétés d'un espace quadridimensionel euclidien ou pseudo-euclidien*, doctoral thesis, K.U.Leuven, 1993.

[14] I.V. Guadalupe and L. Rodriguez : *Normal curvature of surfaces in space forms*, Pacific J. Math. 106, p. 95-103, 1983.

[15] P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *A pointwise inequality in submanifold theory*, Archivum Mathematicum,35, p. 115-128, 1999.