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A NEW COHOMOLOGY ON SPECIAL KINDS OF COMPLEX MANIFOLDS

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Abstract In this paper a new cohomology $H^{p,j,k}$ on complex manifolds admitting flat linear connections is introduced. The new cohomology is constructed using that $\Psi \circ \Psi = O$ for the operator Ψ introduced in [4]. This operator Ψ is a linear algebraic operator and does not contain any differentiation.

1 Introduction

The results of this paper are based on those of [4]. We give them below for the sake of completeness.

Definition 1. Let $\mathcal{V}$ and $\mathcal{V}'$ be vector spaces. For an arbitrary mapping $f : \mathcal{V}^{k-1} \to \mathcal{V}'$ $(k > 1)$ we define a mapping $\Psi f : \mathcal{V}^k \to \mathcal{V}'$ by

$$(\Psi f)(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{k-1}, \mathbf{X}_k) = (-1)^{k-1} f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_{k-1})$$

$$-f(\mathbf{X}_2, \mathbf{X}_3, \cdots, \mathbf{X}_k) + \sum_{i=1}^{k-1} (-1)^{i+1} f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_i + \mathbf{X}_{i+1}, \cdots, \mathbf{X}_{k-1}, \mathbf{X}_k).$$

If $k = 1$, we define $\Psi f = O$.

Theorem 1. *For an arbitrary mapping $f : \mathcal{V}^{k-1} \to \mathcal{V}'$ it holds*

$$(\Psi \circ \Psi) f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k, \mathbf{X}_{k+1}) = O.$$

Note that $\Psi f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k) = O$ if f is a linear mapping.

Theorem 2. *The general differentiable solution of the operator equation*

$$\Psi f(\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_k, \mathbf{X}_{k+1}) = O$$

in the set of differentiable functions $\varphi : \mathcal{V}^{k-1} \to \mathcal{V}'$ $(k \geq 2)$ *is given by*

$$f(\mathbf{X}_1, \cdots, \mathbf{X}_k) = (\Psi F)(\mathbf{X}_1, \cdots, \mathbf{X}_k) + L(\mathbf{X}_1, \cdots, \mathbf{X}_k),$$

for an arbitrary differentiable function $F : \mathcal{V}^{k-1} \to \mathcal{V}'$ *and an arbitrary linear mapping* $L : \mathcal{V}^k \to \mathcal{V}'$ $(k \geq 2)$.

Example. If $k = 4$, the operator equation takes the form

$$(\Psi f)(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) = O,$$

i.e. an explicit form will be given by the functional equation

$$f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) - f(\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) + f(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) -$$

$$f(\mathbf{X}_1, \mathbf{X}_2 + \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5) + f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 + \mathbf{X}_4, \mathbf{X}_5) - f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4 + \mathbf{X}_5) = O.$$

The general differentiable solution of this functional equation is given by

$$f(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) = (\Psi F)(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) + L(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4)$$

$$= F(\mathbf{X}_1 + \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4) + F(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 + \mathbf{X}_4) - F(\mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4)$$

$$-F(\mathbf{X}_1, \mathbf{X}_2 + \mathbf{X}_3, \mathbf{X}_4) - F(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) + L(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4),$$

where F is an arbitrary (differentiable) function and L is an arbitrary linear mapping.

These results are used in the recent paper [5] to construct a cohomology of real manifolds admitting flat linear connections. Using the Theorem 1, in this paper we construct a cohomology on complex manifolds admitting flat linear connections, which is characteristic only for the complex manifolds.

2 A new cohomology on complex manifolds admitting flat linear connections

Let M_n be a complex manifold with complex dimension n which admits a flat linear connection ∇, i.e. with vanishing curvature tensor. We denote by T_x the tangent space at the point $x \in M_n$. In this section we introduce (i,j,k)-forms and the corresponding cohomology groups $H^{i,j,k}$. We denote by $C_k^j(x)$ the vector space of linear mappings $\varphi : (T_x)^k \to (T_x)^j$.

Definition 3. If for each $x \in M_n$, $f(x)$ is a mapping $f(x) : (T_x)^i \to C_k^j(x)$ such that

(i) $f(x)$ is an analytic mapping for each $x \in M_n$,

(ii) $f(x)$ depends analytically on x,

(iii) $\nabla_{\mathbf{X}}(f(\mathbf{X}_1, \cdots, \mathbf{X}_i)) = \mathbf{O}$ if $\nabla_{\mathbf{X}}(\mathbf{X}_1) = \cdots = \nabla_{\mathbf{X}}(\mathbf{X}_i) = \mathbf{O}$, (1)

then f is called an *analytic parallel* (i,j,k)*-form* on M_n.

We assume the convention to say analytic (i,j,k)-form or just (i,j,k)-form instead of analytic parallel (i,j,k)-form. The following lemma follows from the previous definition.

Lemma. *If f is an analytic (i,j,k)-form on M_n, then Ψf is an analytic $(i+1,j,k)$-form on M_n.*

Let $Z^{p,j,k}(M_n)$ be the vector space of closed (p,j,k)-forms, i.e.

$$Z^{p,j,k}(M_n) = \{f : f \text{ is a } (p,j,k)- \text{ form and } \Psi f = \mathbf{O}\}, \tag{2}$$

and $B^{p,j,k}(M_n)$ be the vector space of exact (p,j,k)-forms, i.e.

$$B^{p,j,k}(M_n) = \{\Psi f : f \text{ is a } (p-1,j,k)- \text{ form on } M_n\}. \tag{3}$$

According to Theorem 1, $B^{p,j,k}(M_n)$ is a vector subspace of $Z^{p,j,k}(M_n)$, and two closed (p,j,k)-forms f and g are said to be *congruent* if $f-g$ is an exact (p,j,k)-form. The set of the equivalence classes determines a cohomology group which will be denoted by $H^{p,j,k}$.

Note that the groups $H^{p,j,k}$ are well defined for a chosen flat linear connection ∇. We do not know whether the groups $H^{p,j,k}$ are invariant under the choice of the flat connection. We have assumed that the manifold is flat in order to define *parallel forms.* If we omit the condition (iii) which uses the flat connection, then the corresponding cohomology group may happen to have infinite dimension.

To the end of this section we will consider a special case. Indeed, let the manifold M_n be an affine complex manifold, i.e. it admits an atlas of coordinate neighbourhoods with Jacobi matrices with constant elements. Obviously M_n admits flat linear connection because we can define connection with zero components in any atlas of coordinate neighbourhoods of the above type. Thus the cohomology groups here are defined. We give now an alternative description of the cohomology groups $H^{p,j,k}$ in this special case.

Assume that $\{(U_\alpha, \varphi_\alpha)\}$ is such an atlas of charts that all elements of the corresponding Jacobi matrices are constant functions. For any such chart $(U_\alpha, \varphi_\alpha)$ we consider the vector fields $\mathbf{Z}_1^{(\alpha)}, J(\mathbf{Z}_1^{(\alpha)}), \cdots, \mathbf{Z}_n^{(\alpha)}, J(\mathbf{Z}_n^{(\alpha)})$ tangent to the parametric curves $u_1, v_1, \cdots, u_n, v_n$, where $z_1 = u_1 + iv_1, \cdots, z_n = u_n + iv_n$. It is easy to prove that for any vector field $\mathbf{Z}$ on U_α, $[\mathbf{Z}_1^{(\alpha)}, \mathbf{Z}] = [J(\mathbf{Z}_1^{(\alpha)}), \mathbf{Z}] = \cdots = [\mathbf{Z}_n^{(\alpha)}, \mathbf{Z}] = [J(\mathbf{Z}_n^{(\alpha)}), \mathbf{Z}] = \mathbf{O}$, if and only if $\mathbf{Z}$ is a linear combination of $\mathbf{Z}_1^{(\alpha)}, J(\mathbf{Z}_1^{(\alpha)}), \cdots, \mathbf{Z}_n^{(\alpha)}, J(\mathbf{Z}_n^{(\alpha)})$ with constant coefficients. Hence for any such two overlapping coordinate neighborhoods U_α and U_β it holds

$$[\mathbf{Z}_i^{(\alpha)}, \mathbf{Z}_j^{(\beta)}] = [J(\mathbf{Z}_i^{(\alpha)}), \mathbf{Z}_j^{(\beta)}] = [\mathbf{Z}_i^{(\alpha)}, J(\mathbf{Z}_j^{(\beta)})] = [J(\mathbf{Z}_i^{(\alpha)}), J(\mathbf{Z}_j^{(\beta)})] = \mathbf{O}, \quad (4)$$

$$(1 \leq i, j \leq n).$$

We will denote by $\mathcal{L}$ the Lie derivative and we will use the chosen collection of $2n$ vector fields on M_n for each neighborhood (U_α) and satisfying the property (4). Using the previously introduced flat linear connection on M_n, the parallel analytic (i, j, k) forms are characterized by Definition 3, where the condition (iii) takes the form

(iii) for each α

$$\mathcal{L}_{\mathbf{Z}^{(\alpha)}}(f(\mathbf{Z}_1, \cdots, \mathbf{Z}_i)) = \mathcal{L}_{J(\mathbf{Z}^{(\alpha)})}(f(\mathbf{Z}_1, \cdots, \mathbf{Z}_i)) = \mathbf{O}, \quad (5)$$

if

$$[\mathbf{Z}^{(\alpha)}, \mathbf{Z}_1] = [J(\mathbf{Z}^{(\alpha)}), \mathbf{Z}_1] = \cdots = [\mathbf{Z}^{(\alpha)}, \mathbf{Z}_i] = [J(\mathbf{Z}^{(\alpha)}), \mathbf{Z}_i] = \mathbf{O}. \quad (6)$$

Note that the operator Ψ looks identical to the usual coboundary operator defined in the theory of cohomology of groups with corfficients in a trivial G-module, and more generally with the coboundary operator of Hochschild cohomology of associative algebras ([1], p. 73). Another similarity can be found also in [2].

Now we will consider an example.

Example. Let M be the complex torus T^1 and let us choose an arbitrary analytic tangent vector field $\mathbf{U}$ which is non-zero at each point. Then each vector field $\mathbf{Z}$ is given by $\mathbf{Z} = \alpha\mathbf{U} + \beta J(\mathbf{U})$ and $[\mathbf{Z}, \mathbf{U}] = [\mathbf{Z}, J(\mathbf{U})] = \mathbf{O}$ if and only if α and β are constants. Since $B^{0,j,0}$ is an empty set, $H^{0,j,0} = Z^{0,j,0}$. Further $\Psi f = \mathbf{O}$ and $\mathcal{L}_{\mathbf{U}} f = \mathcal{L}_{J(\mathbf{U})} f = \mathbf{O}$ if and only if f is a complex constant for $j = 0$ and $f = \alpha\mathbf{U} + \beta J(\mathbf{U})$ for $j = 1$. Hence $H^{0,j,0} = \mathbf{C}^1$ for $j \in \{0, 1\}$. Moreover, $H^{0,j,0} = \mathbf{C}^1$ for $j \geq 0$.

Further let $f(z) : T_z \to (T_z)^j$ be such that $\Psi f = \mathbf{O}$, i.e. $f(\mathbf{Z}_1 + \mathbf{Z}_2) = f(\mathbf{Z}_1) + f(\mathbf{Z}_2)$ and hence $f(\alpha\mathbf{Z}) = \alpha f(\mathbf{Z})$. If $j = 0$, then from $\mathbf{O} = \mathcal{L}_{\mathbf{U}}(f(\mathbf{U})) = \mathbf{U}(f(\mathbf{U}))$ and $\mathbf{O} = \mathcal{L}_{J(\mathbf{U})}(f(\mathbf{U})) = J(\mathbf{U})(f(\mathbf{U}))$ it follows that $f(\mathbf{U}) = a = \text{const}$. Analogously $f(J(\mathbf{U})) = b = \text{const}$. Hence we obtain that $H^{1,0,0} = \mathbf{C}$. If $j = 1$, then $f(\mathbf{U}) = a \cdot \mathbf{U} + b \cdot J(\mathbf{U})$ and (iii) implies

$$\mathbf{O} = \mathcal{L}_{\mathbf{U}}(f(\mathbf{U})) = [\mathbf{U}, f(\mathbf{U})] = [\mathbf{U}, a \cdot \mathbf{U} + b \cdot J(\mathbf{U})] = \mathbf{U}(a) \cdot \mathbf{U} + \mathbf{U}(b) \cdot J(\mathbf{U}),$$

$$\mathbf{O} = \mathcal{L}_{J(\mathbf{U})}(f(\mathbf{U})) = [J(\mathbf{U}), f(\mathbf{U})] = [J(\mathbf{U}), a \cdot \mathbf{U} + b \cdot J(\mathbf{U})] =$$
$$= J(\mathbf{U})(a) \cdot \mathbf{U} + J(\mathbf{U})(b) \cdot J(\mathbf{U})$$

and hence a and b are constants over T^1. Thus $H^{1,1,0} = \mathbf{C}$. Note that also $H^{1,j,0} = \mathbf{C}$ for $j > 1$.

Remark. Note that for the s-dimensional torus T^s it holds

$$H^{0,0,0} = \mathbf{C}, \quad H^{0,1,0} = \mathbf{C}^s, \quad H^{1,0,0} = \mathbf{C}^s \quad \text{and} \quad H^{1,1,0} = \mathbf{C}^{s^2}.$$

These cases are simple because there exist global linearly independent vector fields

$$\mathbf{Z}_1, J(\mathbf{Z}_1), \cdots, \mathbf{Z}_n, J(\mathbf{Z}_n)$$

such that $[\mathbf{Z}_p, \mathbf{Z}_q] = [J(\mathbf{Z}_p), \mathbf{Z}_q] = [J(\mathbf{Z}_p), J(\mathbf{Z}_q)] = \mathbf{O}$ for $1 \leq p, q \leq n$.

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