

**CR SUBMANIFOLDS OF MAXIMAL CR DIMENSION
IN COMPLEX PROJECTIVE SPACE
AND ITS HOLOMORPHIC SECTIONAL CURVATURE**

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Abstract. Let M be a CR submanifold of maximal CR dimension in a complex projective space such that the distinguished vector field ξ is parallel with respect to the normal connection. In this article we treat the special case when the shape operator with respect to this vector field has exactly two distinct eigenvalues and we give another sufficient condition for M to be an open subset of a geodesic sphere by discussing its holomorphic sectional curvature.

1. Introduction

Many differential geometers have investigated hypersurfaces of real and complex space forms with constant principal curvatures. Much of the work has involved finding sufficient conditions for a hypersurface to be one of the “standard examples”, characterized by the fact that they have one or two distinct constant principal curvatures. Especially, real hypersurfaces in complex space forms are equipped with a distinguished tangent vector field JN obtained by applying the complex structure J to the unit normal field N , and it was found that computations were more tractable when JN is a principal vector field. Further, it was observed that JN is principal for all homogeneous hypersurfaces in complex projective space. Moreover, geometric characterizations of this property were found and hypersurfaces that satisfy it are now called Hopf hypersurfaces [3], [6], [13].

On the other hand, it is well-known that the holomorphic sectional curvature is an important invariant when investigating the differential geometric properties of Kähler

manifolds. Especially, complex space forms, which have constant holomorphic sectional curvature, are the most fundamental examples among the Kähler manifolds. In [7] the author considered an analogous invariant with respect to the real hypersurfaces in complex projective space and obtained the classification of all these hypersurfaces such that this invariant is constant. Above all, it was proved that when this constant is greater than 4, this hypersurface is an open subset of a geodesic hypersphere.

In this article we consider a similar problem by studying one class of CR submanifolds of maximal CR dimension in complex projective space. Namely, let M^n be a real submanifold of the complex manifold $(\overline{M}^{n+p}, \overline{g})$ with complex structure J . If, for any $x \in M$, the tangent space $T_x(M)$ of M at x satisfies $\dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = n - 1$, then M is called a CR submanifold of maximal CR dimension. Therefore it follows that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus \text{span}\{\xi_x\}$, for any $x \in M$. A real hypersurface is a typical example of CR submanifold of maximal CR dimension and the generalization of some results which are valid for real hypersurfaces to CR submanifolds of maximal CR dimension may be expected, see for example [4]. This paper is devoted to the study of CR submanifolds of maximal CR dimension whose shape operator with respect to ξ has exactly two distinct eigenvalues and whose holomorphic sectional curvature $g(R(X, FX)FX, X)$ is constant, where R denotes the Riemannian curvature tensor of M and F is the skew-symmetric endomorphism acting on $T(M)$. In section 2 we recall some general preliminary facts concerning CR submanifolds and in section 3 we prove the

Main Theorem. *Let M be a connected n -dimensional ($n > 2p - 1$, $p \geq 2$) CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $P^{\frac{n+p}{2}}(\mathbb{C})$ such that the distinguished normal vector field ξ is parallel with respect to the normal connection. If the shape operator with respect to the distinguished normal vector field ξ has exactly two distinct eigenvalues and if the holomorphic sectional curvature $g(R(X, FX)FX, X)$ is constant, then M is an open subset of a geodesic sphere.*

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2. CR submanifolds of maximal CR dimension of complex space forms

Let \overline{M} be an $(n+p)$ -dimensional Kähler manifold with Kähler structure (J, \overline{g}) and let M be an n -dimensional real submanifold of \overline{M} with the immersion ι of M into \overline{M} . Then the tangent bundle $T(M)$ is identified with a subbundle of $T(\overline{M})$ and a Riemannian metric g of M is induced from the Riemannian metric \overline{g} of \overline{M} in such a way that $g(X, Y) = \overline{g}(\iota X, \iota Y)$ where $X, Y \in T(M)$ while we denote also by ι the differential of the immersion. The normal bundle $T^\perp(M)$ is the subbundle of $T(\overline{M})$ consisting of all $\overline{X} \in T(\overline{M})$ which are orthogonal to $T(M)$ with respect to Riemannian metric \overline{g} .

Further, let the maximal J -invariant subspace of the tangent space $T_x(M)$ at $x \in M$, called the holomorphic tangent space at x , has constant dimension for any $x \in M$. Then the submanifold M is called the Cauchy-Riemann submanifold or briefly CR

submanifold and the constant complex dimension of the holomorphic tangent space is called the CR dimension of M [11], [17]. Now, let M be a CR submanifold of maximal CR dimension, that is, at each point x of M , the tangent space $T_x(M)$ satisfies $\dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = n - 1$. We note that in this case the above-given definition of CR submanifolds coincides with the definition of CR submanifolds given by Bejancu in [1]. We refer to [5] for more details and examples of CR submanifolds of maximal CR dimension. Moreover, then it follows that M is odd-dimensional and that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus \text{span}\{\xi_x\}$, for any $x \in M$. Hence, for any tangent vector field X , choosing a local orthonormal basis $\xi, \xi_1, \dots, \xi_{p-1}$ of vectors normal to M , we have the following decomposition into tangential and normal components:

$$(2.1) \quad J\iota X = \iota FX + u(X)\xi,$$

$$(2.2) \quad J\xi = -\iota U + P\xi,$$

$$(2.3) \quad J\xi_a = -\iota U_a + P\xi_a \quad (a = 1, \dots, p - 1),$$

where F and P are skew-symmetric endomorphisms acting on $T(M)$ and $T^\perp(M)$, respectively, $U, U_a, a = 1, \dots, p - 1$ are tangent vector fields and u is one form on M . Moreover, using (2.1), (2.2) and (2.3), the Hermitian property of J implies

$$(2.4) \quad g(U, X) = u(X), \quad U_a = 0 \quad (a = 1, \dots, p - 1),$$

$$(2.5) \quad F^2 X = -X + u(X)U,$$

$$(2.6) \quad u(FX) = 0, \quad FU = 0, \quad P\xi = 0.$$

and therefore, relations (2.2) and (2.3) may be written in the form

$$(2.7) \quad J\xi = -\iota U, \quad J\xi_a = P\xi_a \quad (a = 1, \dots, p - 1).$$

Bringing into use that $\{\eta \in T^\perp(M), \eta \perp \xi\}$ is J -invariant, from now on we denote the orthonormal basis of $T^\perp(M)$ by $\xi, \xi_1, \dots, \xi_q, \xi_{1^*}, \dots, \xi_{q^*}$, where $\xi_{a^*} = J\xi_a$ and $q = \frac{p-1}{2}$. Further, let us denote by $\bar{\nabla}$ and ∇ the Riemannian connection of \bar{M} and M , respectively, and by D the normal connection induced from $\bar{\nabla}$ in the normal bundle of M . They are related by the following well-known Gauss and Weingarten equations

$$(2.8) \quad \bar{\nabla}_{\iota X} \iota Y = \iota \nabla_X Y + h(X, Y),$$

$$(2.9) \quad \bar{\nabla}_{\iota X} \xi = -\iota AX + \sum_{a=1}^q \{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\},$$

$$(2.10) \quad \bar{\nabla}_{\iota X} \xi_a = -\iota A_a X - s_a(X)\xi + \sum_{b=1}^q \{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\},$$

$$(2.11) \quad \bar{\nabla}_{\iota X} \xi_{a^*} = -\iota A_{a^*} X - s_{a^*}(X)\xi + \sum_{b=1}^q \{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\},$$

for all $X, Y \in T(M)$, where h denotes the second fundamental form, A, A_a, A_{a^*} denote the shape operators for the normals ξ, ξ_a, ξ_{a^*} , respectively, and s 's are the coefficients of the normal connection D .

Next, if the ambient manifold is a Kaehler manifold, then $\bar{\nabla}J = 0$ and, using (2.10) and (2.11), it follows that

$$(2.12) \quad A_{a^*}X = FA_aX - s_a(X)U,$$

$$(2.13) \quad s_{a^*}(X) = u(A_aX) = g(A_aX, U) = g(A_aU, X),$$

$$(2.14) \quad s_{a^*b^*} = s_{ab}, \quad s_{a^*b} = -s_{ab^*},$$

$$(2.15) \quad h(X, Y) = g(AX, Y)\xi + \sum_{a=1}^q \{g(A_aX, Y)\xi_a + g(A_{a^*}X, Y)\xi_{a^*}\},$$

for all vectors X, Y tangent to M . Moreover, differentiating relations (2.1) and (2.2) covariantly and comparing the tangential and normal parts, using relation (2.7), we get

$$(2.16) \quad (\nabla_Y F)X = u(X)AY - g(AY, X)U,$$

$$(2.17) \quad (\nabla_Y u)(X) = g(FAY, X),$$

$$(2.18) \quad \nabla_X U = FAX.$$

Further, assuming that the vector field ξ is parallel with respect to the normal connection D , it follows that $D_X\xi = \sum_{a=1}^q \{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\} = 0$, from which $s_a = s_{a^*} = 0$ ($a = 1, \dots, q$). Now, using relations (2.12) and (2.13) we obtain

$$(2.19) \quad A_{a^*} = FA_a \quad (a = 1, \dots, q),$$

$$(2.20) \quad A_a U = 0 \quad (a = 1, \dots, q).$$

Since the second fundamental form $h(X, Y)$ is symmetric with respect to X, Y , (2.15) and (2.19) imply that $FA_{a^*}, a = 1, \dots, q$ are symmetric and hence it follows

$$(2.21) \quad FA_a + A_a F = 0, \quad FA_{a^*} + A_{a^*} F = 0, \quad (a = 1, \dots, q).$$

Finally, if the ambient manifold \bar{M} is a complex space form, i.e. a Kähler manifold of constant holomorphic sectional curvature $4k$, then the curvature tensor \bar{R} of \bar{M} has a special form and the Gauss equation becomes

$$(2.22) \quad \begin{aligned} R(X, Y)Z &= k \{g(Y, Z)X - g(X, Z)Y + g(FY, Z)FX - g(FX, Z)FY \\ &\quad - 2g(FX, Y)FZ\} + g(AY, Z)AX - g(AX, Z)AY \\ &\quad + \sum_{a=1}^q \{g(A_a Y, Z)A_a X - g(A_a X, Z)A_a Y \\ &\quad + g(FA_a Y, Z)FA_a X - g(FA_a X, Z)FA_a Y\}, \end{aligned}$$

for all X, Y, Z tangent to M , where R denotes the Riemannian curvature tensor of M ([9]).

3. A characterization of a geodesic sphere

In this section, we prove the Main theorem. We begin by preparing, without proofs, the following results important for later consideration. To that purpose, let us assume that M is an n -dimensional, $n \geq 3$, CR submanifold of CR dimension $\frac{n-1}{2}$ of $(n+p)$ -dimensional Kähler manifold \bar{M} of constant holomorphic sectional curvature $4k$, such that the distinguished vector field ξ is parallel with respect to the normal connection. Then, if the shape operator A with respect to ξ has only one eigenvalue, it follows that $k = 0$ ([4, Lemma 3.3.]). Moreover, if $k \neq 0$ and the shape operator A has exactly two distinct eigenvalues, then U is an eigenvector of A ([4, Lemma 3.4.]). Especially, if \bar{M} is the complex projective space $P^{\frac{n+p}{2}}(\mathbb{C})$ with constant holomorphic sectional curvature $4k$, $k > 0$ and $n > 3$, then if the shape operator A has exactly two distinct eigenvalues, it follows that they are constant ([4, Lemma 4.1.]). Moreover, for $n > 2p-1$, $p \geq 2$, the multiplicity of the eigenvalue μ corresponding to the eigenvector U of the shape operator A is one ([4, Lemma 4.5.]). If we further suppose that the complex projective space $P^{\frac{n+p}{2}}(\mathbb{C})$ is equipped with Fubini-Study metric of constant holomorphic sectional curvature 4, denoting the second eigenvalue of A by λ and the corresponding eigenvector by X , we may write

$$(3.1) \quad AX = \lambda X + (\mu - \lambda)u(X)U.$$

Furthermore, let us recall that although there are no umbilic hypersurfaces in complex projective space, one sheet of the focal set of a geodesic hypersphere is precisely its center and T. E. Cecil and P. J. Ryan proved

Theorem A. [3] *Let M be a connected real hypersurface in a complex projective space, with at most two distinct principal curvatures at each point. Then M is an open subset of a geodesic hypersphere.*

This theorem was first proved by R. Takagi [16], under the additional condition that the principal curvatures are constant.

We now consider the holomorphic sectional curvature $g(R(X, FX)FX, X)$ of the CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $P^{\frac{n+p}{2}}(\mathbb{C})$ equipped with Fubini-Study metric of constant holomorphic sectional curvature 4. Namely, using the Gauss equation (2.22) and relation (2.21), a straightforward computation yields

$$(3.2) \quad g(R(X, FX)FX, X) = 4 + g(AX, X)g(AFX, FX) - g(FX, AX)^2 - 2\|k(X, X)\|^2,$$

where $k : T_0(M) \times T_0(M) \rightarrow \text{span}\{\xi_1, \dots, \xi_q, \xi_{1^*}, \dots, \xi_{q^*}\}$ is a symmetric bilinear form defined by

$$k(X, Y) = h(X, Y) - g(AX, Y)\xi = \sum_{a=1}^q \{g(A_a X, Y)\xi_a + g(A_{a^*} X, Y)\xi_{a^*}\}$$

and $T_0(M) = \{X \in T(M) | g(X, U) = 0\}$.

Now, let us prove the following

Lemma 3.1. *Let M be a connected n -dimensional ($n > 2p - 1, p \geq 2$) CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $P^{\frac{n+p}{2}}(\mathbb{C})$ such that the distinguished normal vector field ξ is parallel with respect to the normal connection. If the shape operator with respect to the distinguished normal vector field ξ has exactly two distinct eigenvalues and if the holomorphic sectional curvature $g(R(X, FX)FX, X)$ is constant, then $A_a = A_{a^*} = 0, a = 1, \dots, q$, where A_a, A_{a^*} are the shape operators for the normals ξ_a, ξ_{a^*} , respectively.*

Proof: Using the above notations, relations (3.1) and (3.2) imply

$$(3.3) \quad g(R(X, FX)FX, X) = 4 + \lambda^2 - 2\|k(X, X)\|^2.$$

From the hypothesis that $g(R(X, FX)FX, X)$ and λ are constant, it follows that $\|k(X, X)\|$ is constant, too.

Furthermore, we note that F is an almost complex structure on $T_0(M)$. Since the subbundle $T_1^\perp(M) = \{\eta \in T^\perp(M) | \bar{g}(\eta, \xi) = 0\}$ of the normal bundle $T^\perp(M)$ is J -invariant, using (2.1), (2.7) and the Gauss equation (2.8), it follows that

$$(3.4) \quad Jk(X, Y) = k(X, FY),$$

which shows that the bilinear form k is almost complex. The discriminant Δ of k for a plane $span\{X, Y\}$ is given by

$$\Delta_{XY} = \frac{\bar{g}(k(X, X), k(Y, Y)) - \|k(X, Y)\|^2}{g(X, X)g(Y, Y) - g(X, Y)^2}.$$

Then, by (3.4), for a unit vector field X belonging to $T_0(M)$, we have $\Delta_{XFX} = -2\|k(X, X)\|^2$. We note that Δ_{XFX} is just corresponding to the holomorphic difference Δ_{hol} in [14].

Now, we bring into use O’Neil’s results proved in [14]. Namely, since $\|k(X, X)\|$ is constant, it follows that the discriminant Δ_{XFX} is constant and using Lemma 6. [14], it follows that k is isotropic. Finally, using Lemma 8. [14], we conclude that $\frac{n-1}{2} (\frac{n-1}{2} + 1) = \frac{n^2-1}{4}$ vectors $k(e_i, e_j), Jk(e_i, e_j)$ are orthogonal, where $e_1, \dots, e_{\frac{n-1}{2}}, Je_1, \dots, Je_{\frac{n-1}{2}}$ is an orthonormal basis for $T_0(M)$. However, since $p < \frac{n+1}{2}$, it follows that $\frac{n^2-1}{4} > p-1$ and therefore $k(e_i, e_j) = 0$, i.e. $h(X, Y) = g(AX, Y)\xi$, or, equivalently, $A_a = 0, A_{a^*} = 0, a = 1, \dots, q$. \square

Making use of this result, we prove

Theorem 3.2. *Let M be an n -dimensional ($n > 2p - 1, p \geq 2$) CR submanifold of CR dimension $\frac{n-1}{2}$ of a complex projective space $P^{\frac{n+p}{2}}(\mathbb{C})$ such that the distinguished normal vector field ξ is parallel with respect to the normal connection. If the*

shape operator with respect to the distinguished normal vector field ξ has exactly two distinct eigenvalues and if the holomorphic sectional curvature $g(R(X, FX)FX, X)$ is constant, then there exists a real $n+1$ -dimensional totally geodesic complex projective space $P^{\frac{n+1}{2}}(\mathbb{C})$ such that $M \subset P^{\frac{n+1}{2}}(\mathbb{C})$.

Proof: First, let us define $N_0(x) = \{\xi \in T_x^\perp(M) | A_\xi = 0\}$ and let $H_0(x)$ be the maximal J -invariant subspace of $N_0(x)$, that is, $H_0(x) = JN_0(x) \cap N_0(x)$. Then, using Lemma 3.1., it follows that $N_0(x) = \text{span}\{\xi_1(x), \dots, \xi_q(x), \xi_{1^*}(x), \dots, \xi_{q^*}(x)\}$. Moreover, by the second equation of (2.7), we obtain $JN_0(x) = N_0(x)$ and consequently $H_0(x) = \text{span}\{\xi_1(x), \dots, \xi_q(x), \xi_{1^*}(x), \dots, \xi_{q^*}(x)\}$. Hence the orthogonal complement $H_1(x)$ of $H_0(x)$ in $T^\perp(M)$ is spanned by ξ . Further, since ξ is parallel with respect to the normal connection, we can apply the codimension reduction theorem for real submanifolds of complex projective space ([12]) and conclude that there exists a real $(n+1)$ -dimensional totally geodesic complex projective space of $P^{\frac{n+p}{2}}(\mathbb{C})$ such that M is its real hypersurface. \square

Therefore, we can apply the results of real hypersurface theory and prove the Main theorem. Namely, using Theorem 3.2., the submanifold M can be regarded as a real hypersurface of $P^{\frac{n+1}{2}}(\mathbb{C})$ which is a totally geodesic submanifold in $P^{\frac{n+p}{2}}(\mathbb{C})$. In what follows we denote $P^{\frac{n+1}{2}}(\mathbb{C})$ by M' and by ι_1 the immersion of M into M' and by ι_2 the totally geodesic immersion of M' into $P^{\frac{n+p}{2}}(\mathbb{C})$. Then, from the Gauss equation (2.8), it follows that

$$\nabla'_{\iota_1 X} \iota_1 Y = \iota_1 \nabla_X Y + g(A'X, Y)\xi',$$

where A' is the corresponding shape operator and ξ' is a unit normal vector field to M in M' . Consequently, by using the Gauss equation and $\iota = \iota_2 \cdot \iota_1$, we derive

$$(3.5) \quad \bar{\nabla}_{\iota_2 \cdot \iota_1 X} \iota_2 \cdot \iota_1 Y = \iota_2 \nabla'_{\iota_1 X} \iota_1 Y + \bar{h}(\iota_1 X, \iota_1 Y) = \iota_2(\iota_1 \nabla_X Y + g(A'X, Y)\xi'),$$

since M' is totally geodesic in $P^{\frac{n+p}{2}}(\mathbb{C})$. Further, comparing relation (3.5) with relation (2.8), it follows that $\xi = \iota_2 \xi'$ and $A = A'$. As M' is a complex submanifold of $P^{\frac{n+p}{2}}(\mathbb{C})$ with the induced complex structure J' , we have $J\iota_2 X' = \iota_2 J'X'$, $X' \in T(M')$. Thus, from (2.1) it follows that

$$J\iota X = \iota_2 J' \iota_1 X = \iota F' X + \nu'(X)\iota_2 \xi' = \iota F' X + \nu'(X)\xi$$

and therefore, we conclude that $F = F'$ and $\nu' = \nu$.

Finally, since in Theorem 3.2. we established that M , with exactly two distinct eigenvalues of the shape operator A with respect to the distinguished vector field ξ , is a real hypersurface of $P^{\frac{n+1}{2}}(\mathbb{C})$, we prove the Main theorem by using the well known Theorem A cited at the beginning of this section.

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