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THE APPLICATION OF LIOUVILLE VECTOR FIELDS IN Osc^3M

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Abstract. R. Miron and Gh. Atanasiu studied the geometry of Osc^kM. Among many various problems which was solved, they introduced the adapted basis, the d-connection and gave its curvature theory. Different structures as almost product structure, metric structure was determined and the spray theory was given.

Here the attention on $E = Osc^3M$ will be restricted. In Osc^3M the Liouville vector fields have important role at definition of almost contact structure J and using J and the Liouville vector fields the sprays are defined. The geodesic lines are integral curves of sprays. The Zermello's conditions which give the independence of the integral of action from the parametrization of the curve are also expressed by the Liouville vector fields.

Almost all the results obtained here can be found in Miron's book [19], [20], even for the space Osc^kM, but here the transformation group is slightly different and the methods of some proofs are new.

1 Adapted basis in $T(Osc^3M)$ and $T^*(Osc^3M)$

Let $E = Osc^3M$ be a $4n$ dimensional C^∞ manifold. In some local chart (U, φ) some point $u \in E$ has coordinates

$$(x^a, y^{1a}, y^{2a}, y^{3a}) = (y^{0a}, y^{1a}, y^{2a}, y^{3a}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \ldots = 0, 1, 2, 3.$$

If in some other chart (U', φ') the point $u \in E$ has coordinates $(x^{a'}, y^{1a'}, y^{2a'}, y^{3a'})$, then in $U \cap U'$ the allowable coordinate transformation are given by:

$$\begin{aligned}
&\text{(a)} \quad x^{a'} = x^{a'}(x^1, x^2, \ldots, x^n) \\
&\text{(b)} \quad y^{1a'} = \frac{\partial x^{a'}}{\partial x^a} y^{1a} = \frac{\partial y^{0a'}}{\partial y^{0a}} y^{1a} \\
&\text{(c)} \quad y^{2a'} = \frac{\partial y^{1a'}}{\partial y^{0a}} y^{1a} + \frac{\partial y^{1a'}}{\partial y^{1a}} y^{2a} \\
&\text{(d)} \quad y^{3a'} = \frac{\partial y^{2a'}}{\partial y^{0a}} y^{1a} + \frac{\partial y^{2a'}}{\partial y^{1a}} y^{2a} + \frac{\partial y^{2a'}}{\partial y^{2a}} y^{3a}.
\end{aligned} \tag{1.1}$$

Some nice example of the space E can be obtained if the points $(x^a) \in M$ $(dimM = n)$ are considered as the points of the curve $x^a = x^a(t)$ and $y^{\alpha a}$, $\alpha = 1, 2, 3$, are defined by

$$y^{1a} = \frac{dx^a}{dt}, \; y^{2a} = \frac{d^2x^a}{dt^2} = \frac{dy^{1a}}{dt}, \; y^{3a} = \frac{d^3x^a}{dt^3} = \frac{dy^{2a}}{dt}.$$

M is the base manifold and $(x^a) \in M$ is the projection of $(x^a, y^{1a}, y^{2a}, y^{3a}) \in E$ on M. In [15], [16] $y^{\alpha a} = \frac{1}{\alpha!}\frac{d^\alpha x^a}{dt^\alpha}$, $\alpha = 1, \ldots, k$ and the transformations (1.1) have different form. If in $U \cap U'$ the equation

$$x^{a'} = x^{a'}(x^1(t), x^2(t), \ldots, (x^n(t))$$

is valid, then it is easy to see that

$$\begin{aligned}
y^{1a'} &= \frac{dx^{a'}}{dt} = y^{1a'}(x^a, y^{1a}), \\
y^{2a'} &= \frac{dy^{1a'}}{dt} = y^{2a'}(x^a, y^{1a}, y^{2a}), \\
y^{3a'} &= \frac{dy^{2a'}}{dt} = y^{3a'}(x^a, y^{1a}, y^{2a}, y^{3a}),
\end{aligned} \tag{1.2}$$

satisfy (1.1b), (1.1c) and (1.1d) respectively and the explicite form of (1.1) is the following:

$$
\begin{aligned}
x^{a'} &= x^{a'}(x^1, x^2, \ldots, x^n) \\
y^{1a'} &= \frac{\partial x^{a'}}{\partial x^a} y^{1a}, \\
y^{2a'} &= \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1a} y^{1b} + \frac{\partial x^{a'}}{\partial x^a} y^{2a}, \\
y^{3a'} &= \frac{\partial^3 x^{a'}}{\partial x^a \partial x^b \partial x^c} y^{1a} y^{1b} y^{1c} + 3 \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1a} y^{2b} + \frac{\partial x^{a'}}{\partial x^a} y^{3a}.
\end{aligned}
\tag{1.3}
$$

Theorem 1.1 *The transformations determined by (1.1) form a pseudogroup.*

With determination of the group of allowable coordinate transformations the first step to construction of some geometry is made. The second important step is the construction of the adapted basis in $T(E)$, which depends on the choice of the coefficients of the nonlinear connections, here denoted by N and M.

The following abbreviations

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}}, \ \alpha = 1, 2, 3, \ \text{and } \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}$$

will be used. From (1.3) it follows

$$
\begin{aligned}
&\partial_{0a} y^{0a'} = \partial_{1a} y^{1a'} = \partial_{2a} y^{2a'} = \partial_{3a} y^{3a'} = \frac{\partial x^{a'}}{\partial x^a} = A_a^{a'}, \\
&\frac{dA_a^{a'}}{dt} = \partial_{0a} y^{1a'} = \frac{1}{2} \partial_{1a} y^{2a'} = \frac{1}{2}\frac{2}{3} \partial_{2a} y^{3a'} = \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1b} = B_a^{a'}, \\
&\frac{dB_a^{a'}}{dt} = \partial_{0a} y^{2a'} = \frac{1}{3} \partial_{1a} y^{3a'} = \frac{\partial^3 x^{a'}}{\partial x^a \partial x^b \partial x^c} y^{1b} y^{1c} + \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{2b} = C_a^{a'}, \\
&\frac{dC_a^{a'}}{dt} = \partial_{0a} y^{3a'} = D_a^{a'}.
\end{aligned}
\tag{1.4}
$$

The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}, \partial_{3a}\} = \{\partial_{\alpha a}\} \tag{1.5}$$

The elements of $\bar{B}$ with respect to (1.1) are not transformed as d-tensors. They satisfy the following relations:

$$
\begin{aligned}
\partial_{0a} &= (\partial_{0a} y^{0a'})\partial_{0a'} + (\partial_{0a} y^{1a'})\partial_{1a'} + (\partial_{0a} y^{2a'})\partial_{2a'} + (\partial_{0a} y^{3a'})\partial_{3a'} \\
\partial_{1a} &= (\partial_{1a} y^{1a'})\partial_{1a'} + (\partial_{1a} y^{2a'})\partial_{2a'} + (\partial_{1a} y^{3a'})\partial_{3a'} \\
\partial_{2a} &= (\partial_{2a} y^{2a'})\partial_{2a'} + (\partial_{2a} y^{3a'})\partial_{3a'} \\
\partial_{3a} &= (\partial_{3a} y^{3a'})\partial_{3a'}.
\end{aligned}
\tag{1.6}
$$

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dx^a, dy^{1a}, dy^{2a}, dy^{3a}\} = \{dy^{\alpha a}\}. \tag{1.7}$$

The elements of $\bar{B}^*$ with respect to (1.1) are transformed in the following way (see (1.2)):

$$\begin{aligned} dx^{a'} &= \frac{\partial x^{a'}}{\partial x^a} dx^a \Leftrightarrow dy^{0a'} = (\partial_{0a} y^{0a'}) dy^{0a} \\ dy^{1a'} &= (\partial_{0a} y^{1a'}) dy^{0a} + (\partial_{1a} y^{1a'}) dy^{1a} \\ dy^{2a'} &= (\partial_{0a} y^{2a'}) dy^{0a} + (\partial_{1a} y^{2a'}) dy^{1a} + (\partial_{2a} y^{2a'}) dy^{2a} \\ dy^{3a'} &= (\partial_{0a} y^{3a'}) dy^{0a} + (\partial_{1a} y^{3a'}) dy^{1a} + (\partial_{2a} y^{3a'}) dy^{2a} + (\partial_{3a} y^{3a'}) dy^{3a}. \end{aligned} \tag{1.8}$$

The adapted basis B^* of $T^*(E)$ is given by:

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\}, \tag{1.9}$$

where

$$\begin{aligned} \delta y^{0a} &= dx^a = dy^{0a} \\ \delta y^{1a} &= dy^{1a} + M^{1a}_{0b} dy^{0b} \\ \delta y^{2a} &= dy^{2a} + M^{2a}_{1b} dy^{1b} + M^{2a}_{0b} dy^{0b} \\ \delta y^{3a} &= dy^{3a} + M^{3a}_{2b} dy^{2b} + M^{3a}_{1b} dy^{1b} + M^{3a}_{0b} dy^{0b}. \end{aligned} \tag{1.10}$$

Theorem 1.2 *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as d-tensor field, i.e.*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{\alpha a}, \; \alpha = 0, 1, 2, 3,$$

are the following equations:

$$\begin{aligned} &(a) \quad M^{1a}_{0b} \partial_{1a} y^{1a'} = M^{1a'}_{0b'} \partial_{0b} y^{0b'} + \partial_{0b} y^{1a'} \\ &(b) \quad M^{2a}_{1b} \partial_{2a} y^{2a'} = M^{2a'}_{1c'} \partial_{1b} y^{1c'} + \partial_{1b} y^{2a'} \\ &(c) \quad M^{2a}_{0b} \partial_{2a} y^{2a'} = M^{2a'}_{0c'} \partial_{0b} y^{0c'} + M^{2a'}_{1c'} \partial_{0b} y^{1c'} + \partial_{0b} y^{2a'} \\ &(d) \quad M^{3a}_{2b} \partial_{3a} y^{3a'} = M^{3a'}_{2c'} \partial_{2b} y^{2c'} + \partial_{2b} y^{3a'} \\ &(e) \quad M^{3a}_{1b} \partial_{3a} y^{3a'} = M^{3a'}_{1c'} \partial_{1b} y^{1c'} + M^{3a'}_{2c'} \partial_{1b} y^{2c'} + \partial_{1b} y^{3a'} \\ &(f) \quad M^{3a}_{0b} \partial_{3a} y^{3a'} = M^{3a'}_{0c'} \partial_{0b} y^{0c'} + M^{3a'}_{1c'} \partial_{0b} y^{1c'} + M^{3a'}_{2c'} \partial_{0b} y^{2c'} + \partial_{0b} y^{3a'}. \end{aligned} \tag{1.11}$$

From (1.11) and (1.4) it follows that (1.11) is a system in which equations of second, third and fourth order appeared, so there are infinity functions

$$\begin{aligned} &M^{1a}_{0b}=M^{1a}_{0b}(x,y^1),\ M^{2a}_{1b}=M^{2a}_{1b}(x,y^1),\ M^{3a}_{2b}=M^{3a}_{2b}(x,y^1),\\ &M^{2a}_{0b}=M^{2a}_{0b}(x,y^1,y^2),\ M^{3a}_{1b}=M^{3a}_{1b}(x,y^1,y^2),\\ &M^{3a}_{0b}=M^{3a}_{0b}(x,y^1,y^2,y^3), \end{aligned} \tag{1.12}$$

which are the solutions of (1.11). From the choise of M depends the adapted basis B^* ((1.9)).

Let us denote the adapted basis of $T(E)$ by B, where

$$B=\{\delta_{0a},\delta_{1a},\delta_{2a},\delta_{3a}\}=\{\delta_{\alpha a}\}, \tag{1.13}$$

and

$$\begin{array}{lcl} \delta_{0a} &=& \partial_{0a} - N^{1b}_{0a}\partial_{1b} - N^{2b}_{0a}\partial_{2b} - N^{3b}_{0a}\partial_{3b},\\ \delta_{1a} &=& \partial_{1a} - N^{2b}_{1a}\partial_{2b} - N^{3b}_{1a}\partial_{3b}\\ \delta_{2a} &=& \partial_{2a} - N^{3b}_{2a}\partial_{3b}\\ \delta_{3a} &=& \partial_{3a}. \end{array} \tag{1.14}$$

Theorem 1.3 *The necessary and sufficient conditions that B ((1.13)) be dual to B^* ((1.9)), (when $\bar{B}$ ((1.5)) is dual to $\bar{B}^*$ ((1.7)) i.e.*

$$< \delta_{\alpha a}\delta y^{\beta b} >= \delta^{\beta}_{\alpha}\delta^{b}_{a}$$

are the following relations:

$$\begin{array}{lcl} N^{1b}_{0a} &=& M^{1b}_{0a}\\ N^{2b}_{0a} &=& M^{2b}_{0a}-M^{2b}_{1c}N^{1c}_{0a}\\ N^{3b}_{0a} &=& M^{3b}_{0a}-M^{3b}_{1c}N^{1c}_{0a}-M^{3b}_{2c}N^{2c}_{0a}\\ N^{2b}_{1a} &=& M^{2b}_{1a}\\ N^{3b}_{1a} &=& M^{3b}_{1a}-M^{3b}_{2c}N^{2c}_{1a}\\ N^{3b}_{2a} &=& M^{3b}_{2a}. \end{array} \tag{1.15}$$

Theorem 1.4 *The necessary and sufficient conditions that $\delta_{\alpha a}$ with respect to (1.1) are transformed as d-tensors, i.e.*

$$\delta_{\alpha a'}=\frac{\partial x^a}{\partial x^{a'}}\delta_{\alpha a},\ \alpha=0,1,2,3, \tag{1.16}$$

are the following formulae:

$$
\begin{aligned}
N^{1b'}_{0a'}\partial_{0a}y^{0a'} &= N^{1c}_{0a}\partial_{1c}y^{1b'} - \partial_{0a}y^{1b'} \\
N^{2b'}_{0a'}\partial_{0a}y^{0a'} &= N^{2c}_{0a}\partial_{2c}y^{2b'} + N^{1c}_{0a}\partial_{1c}y^{2b'} - \partial_{0a}y^{2b'} \\
N^{3b'}_{0a'}\partial_{0a}y^{0a'} &= N^{3c}_{0a}\partial_{3c}y^{3b'} + N^{2c}_{0a}\partial_{2c}y^{3b'} + N^{1c}_{0a}\partial_{1c}y^{3b'} - \partial_{0a}y^{3b'} \\
N^{2b'}_{1a'}\partial_{1a}y^{1a'} &= N^{2c}_{1a}\partial_{2c}y^{2b'} - \partial_{1a}y^{2b'} \\
N^{3b'}_{1a'}\partial_{1a}y^{1a'} &= N^{3c}_{1a}\partial_{3c}y^{3b'} + N^{2c}_{1a}\partial_{2c}y^{3b'} - \partial_{1a}y^{3b'} \\
N^{3b'}_{2a'}\partial_{2a}y^{2a'} &= N^{3b}_{2a}\partial_{3b}y^{3b'} - \partial_{2a}y^{3b'}.
\end{aligned}
\tag{1.17}
$$

From (1.13) and (1.14) it follows

$$
\begin{aligned}
\partial_{3a} &= \delta_{3a} \\
\partial_{2a} &= \delta_{2a} + M^{3b}_{2a}\delta_{3b} \\
\partial_{1a} &= \delta_{1a} + M^{2b}_{1a}\delta_{2b} + M^{3b}_{1a}\delta_{3b} \\
\partial_{0a} &= \delta_{0a} + M^{1b}_{0a}\delta_{1b} + M^{2b}_{0a}\delta_{2b} + M^{3b}_{0a}\delta_{3b}.
\end{aligned}
\tag{1.18}
$$

2 The adapted basis which is conprehensive with J, Liouville vector fields

It is obvious that the introduced transformation group given by (1.1) instead of that introduced by R. Miron [16], [17] results a new adapted basis B ((1.13)) and B^* ((1.9)). These bases are dual to each other, their elements transform as d-vector (or covector) fields, but they are not convenient for the presentation of the almost tangent structure J, for which $J^4 = 0$ and $JT_H = T_{V_1}$, $JT_{V_1} = T_{V_2}$, $JT_{V_2} = T_{V_3}$, $JT_{V_3} = 0$. To obtain such a basis we take:

$$
\begin{aligned}
\delta y^{0a} &= dy^{0a} = dx^a \\
\delta y^{1a} &= dy^{1a} + M^{1a}_{0b}dy^{0b} \\
\delta y^{2a} &= \frac{1}{2}dy^{2a} + M^{2a}_{1b}dy^{1b} + M^{2a}_{0b}dy^{0b} \\
\delta y^{3a} &= \frac{1}{6}dy^{3a} + \frac{1}{2}M^{3a}_{2b}dy^{2b} + M^{3a}_{1b}dy^{1a} + M^{3a}_{0b}dy^{0b}.
\end{aligned}
\tag{2.1}
$$

Theorem 2.1 *The necessary and sufficient conditions that $\delta y^{\alpha a}(\alpha = 0, 1, 2, 3)$ given by (2.1) are transformed as d-tensor fields, are the following equations:*

$$
M^{1a}_{0b}\partial_{0a}y^{0a'} = M^{1a'}_{0b'}\partial_{0b}y^{0b'} + \partial_{0b}y^{1a'} \tag{2.2}
$$

$$
\begin{aligned}
M^{2a}_{1b}\partial_{2a}y^{2a'} &= M^{2a'}_{1b'}\partial_{1b}y^{1b'} + \frac{1}{2}\partial_{1b}y^{2a'} \\
M^{3a}_{2b}\partial_{3a}y^{3a'} &= M^{3a'}_{2b'}\partial_{2b}y^{2b'} + \frac{1}{3}\partial_{2b}y^{3a'} \\
M^{2a}_{0b}\partial_{2a}y^{2a'} &= M^{2a'}_{0b'}\partial_{0b}y^{0b'} + M^{1a'}_{1b'}\partial_{0b}y^{1b'} + \frac{1}{2}\partial_{0b}y^{2a'} \\
M^{3a}_{1b}\partial_{3a}y^{3a'} &= M^{3a'}_{1b'}\partial_{1b}y^{1b'} + \frac{1}{2}M^{3a'}_{2b'}\partial_{1b}y^{2b'} + \frac{1}{6}\partial_{1b}y^{3a'} \\
M^{3a}_{0b}\partial_{3a}y^{3a'} &= M^{3a'}_{0b'}\partial_{0b}y^{0b'} + M^{3a'}_{1b'}\partial_{0b}y^{1b'} + \frac{1}{2}M^{3a'}_{2b'}\partial_{0b}y^{2b'} + \frac{1}{6}\partial_{0b}y^{3a'}.
\end{aligned}
$$

From (1.4) it follows that M^{1a}_{0b}, M^{2a}_{1b} and M^{3a}_{2b} have the same law of transformation, also M^{2a}_{0b} and M^{3a}_{1b} transform in the same way. This fact allows us to take

$$M^{1a}_{0b} = M^{2a}_{1b} = M^{3a}_{2b}, \quad M^{2a}_{0b} = M^{3a}_{1b}. \tag{2.3}$$

If (2.3) is valid the adapted basis

$$B'^{*} = \{\delta' y^{0a}, \delta' y^{1a}, \delta' y^{2a}, \delta' y^{3a}\} \tag{2.4}$$

is given by

$$
\begin{aligned}
\delta' y^{0a} &= dx^a = dy^{0a} \\
\delta' y^{1a} &= dy^{1a} + M^{1a}_{0b}dy^{0b} \\
\delta' y^{2a} &= \frac{1}{2}dy^{2a} + M^{1a}_{0b}dy^{1b} + M^{2a}_{0b}dy^{0b} \\
\delta' y^{3a} &= \frac{1}{6}dy^{3a} + \frac{1}{2}M^{1a}_{0b}dy^{2b} + M^{2a}_{0b}dy^{1b} + M^{3a}_{0b}dy^{0b}.
\end{aligned} \tag{2.5}
$$

Theorem 2.2 *The structure J defined on $T^*(E)$ by*

$$J(dy^{3a}) = 3dy^{2a}, \quad J(dy^{2a}) = 2dy^{1a}, \quad J(dy^{1a}) = dy^{0a}, \quad J(dy^{0a}) = 0 \tag{2.6}$$

is a tensor field of type (1,1), and satisfies the relation $J^4 = 0$.

From (2.6) and (2.5) it follows

$$J(\delta' y^{3a}) = \delta' y^{2a}, \quad J(\delta' y^{2a}) = \delta' y^{1a}, \quad J(\delta' y^{1a}) = \delta' y^{0a}, J(\delta' y^{0a}) = 0. \tag{2.7}$$

Let us denote by

$$B' = \{\delta'_{0a}, \delta'_{1a}, \delta'_{2a}, \delta'_{3a}\}$$

the adapted basis of $T(E)$ given by

$$\begin{array}{lcl} \delta'_{0a} &=& \partial_{0a} - N^{1b}_{0a}\partial_{1b} - 2N^{2b}_{0a}\partial_{2b} - 6N^{3b}_{0a}\partial_{3b} \\ \delta'_{1a} &=& \partial_{1a} - 2N^{1b}_{0a}\partial_{2b} - 6N^{2b}_{0a}\partial_{3b} \\ \delta'_{2a} &=& 2\partial_{2a} - 6N^{1b}_{0a}\partial_{3b} \\ \delta'_{3a} &=& 6\partial_{3a}. \end{array} \tag{2.8}$$

Theorem 2.3 *The adapted basis B' and B'^* are dual to each other if*

$$\begin{array}{l} N^{1b}_{0a} = M^{1b}_{0a}, \\ N^{2b}_{0a} = M^{2b}_{0a} - M^{1b}_{0c}N^{1c}_{0a} \\ N^{3b}_{0a} = M^{3b}_{0a} - M^{2b}_{0c}N^{1c}_{0a} - M^{1b}_{0c}N^{2c}_{0a} \end{array} \tag{2.9}$$

or equivalently

$$\begin{array}{l} M^{1b}_{0a} = N^{1b}_{0a}, \\ M^{2b}_{0a} = N^{2b}_{0a} + N^{1b}_{0c}N^{1c}_{0a} \\ M^{3b}_{0a} = N^{3b}_{0a} + N^{2b}_{0c}N^{1c}_{0a} + N^{1b}_{0c}N^{2c}_{0a} + N^{1c}_{0a}N^{1d}_{0c}N^{1b}_{0d} \end{array} \tag{2.9a}$$

Theorem 2.4 *The elements of basis B' given by (2.8) are transformed as d-tensor fields if*

$$\begin{array}{l} N^{1b'}_{0a'}\partial_{0a}y^{0a'} = N^{1c}_{0a}\partial_{1c}y^{1b'} + \partial_{0a}y^{1b'} \\ N^{2b'}_{0a'}\partial_{0a}y^{0a'} = N^{2c}_{0a}\partial_{2c}y^{2b'} + \dfrac{1}{2}N^{1c}_{0a}\partial_{1c}y^{2b'} - \dfrac{1}{2}\partial_{0a}y^{2b} \\ N^{3b'}_{0a'}\partial_{0a}y^{0a'} = N^{3c}_{0a}\partial_{3c}y^{3b'} + \dfrac{1}{3}N^{2c}_{0a}\partial_{2c}y^{3b'} + \dfrac{1}{6}N^{1c}_{0a}\partial_{1c}y^{3b'} - \dfrac{1}{6}\partial_{0a}y^{3b'}. \end{array} \tag{2.10}$$

Theorem 2.5 *The tensor J considered as a linear transformation on $T^*(E)$ in the basis $\bar{B}$ and $\bar{B}^*$ has the form:*

$$J = dy^{0b}J^{1a}_{0b} \otimes \partial_{1a} + dy^{1b}J^{2a}_{1b} \otimes \partial_{2a} + dy^{2b}J^{3a}_{2b} \otimes \partial_{3a}, \tag{2.11}$$

where

$$J^{1a}_{0b} = \delta^a_b, \quad J^{2a}_{1b} = 2\delta^a_b, \quad J^{3a}_{2b} = 3\delta^a_b,$$

or in the matrix form

$$J = [dy^{0b}dy^{1b}dy^{2b}dy^{3b}] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \otimes \begin{bmatrix} \partial_{0b} \\ \partial_{1b} \\ \partial_{2b} \\ \partial_{3b} \end{bmatrix}. \tag{2.12}$$

The tensor J in the basis B' and B'^* determined by (2.8) and (2.4) has the form

$$J = \delta' y^{0a} \otimes \delta'_{1a} + \delta' y^{1a} \otimes \delta'_{2a} + \delta' y^{2a} \otimes \delta'_{3a}. \tag{2.13}$$

Theorem 2.6 *The tensor J considered as a linear transformation on $T(E)$ in the basis $\bar{B}$ and $\bar{B}^*$ has the form*

$$J = \partial_{1a} \otimes dy^{0a} + 2\partial_{2a} \otimes dy^{1a} + 3\partial_{3a} \otimes dy^{2a} \tag{2.14}$$

and in the basis B' and B'^* has the form

$$J = \delta'_{1a} \otimes \delta' y^{0a} + \delta'_{2a} \otimes \delta' y^{1a} + \delta'_{3a} \otimes \delta' y^{2a}. \tag{2.15}$$

From (2.14) and (2.15) it follows

$$J(\partial_{0a}) = \partial_{1a}, \quad J(\partial_{1a}) = 2\partial_{2a}, \quad J(\partial_{2a}) = 3\partial_{3a}, \quad J(\partial_{3a}) = 0 \tag{2.16}$$

$$J(\delta'_{0a}) = \delta'_{1a}, \quad J(\delta'_{1a}) = \delta'_{2a}, \quad J(\delta'_{2a}) = \delta'_{3a}, \quad J(\delta'_{3a}) = 0. \tag{2.17}$$

Definition 2.1 *With respect to the coordinate transformation (1.3) the Liouville vector fields have the form*

$$\begin{aligned} &\Gamma_{(1)} = y^{1a}\partial_{3a}, \quad \Gamma_{(2)} = y^{1a}\partial_{2a} + 3y^{2a}\partial_{3a}, \\ &\Gamma_{(3)} = y^{1a}\partial_{1a} + 2y^{2a}\partial_{2a} + 3y^{3a}\partial_{3a}. \end{aligned} \tag{2.18}$$

In the geometry where Miron's transformation group is used ([15], [16], [17]) $\Gamma_{(1)}$ and $\Gamma_{(3)}$ are the same as here, but $\Gamma_{(2)} = y^{1a}\partial_{2a} + 2y^{2a}\partial_{3a}$.

The vector fields $\Gamma_{(\alpha)}$, $\alpha = 1, 2, 3$ given by (2.18) in the basis B has the form

$$\begin{aligned} &\Gamma_{(1)} = z_1^{3a}\delta_{3a}, \quad \Gamma_{(2)} = z_2^{2a}\delta_{2a} + z_2^{3a}\delta_{3a}, \\ &\Gamma_{(3)} = z_3^{1a}\delta_{1a} + z_3^{2a}\delta_{2a} + z_3^{3a}\delta_{3a}. \end{aligned} \tag{2.19}$$

The relation between the components is given by:

$$\begin{aligned} &z_1^{3a} = y^{1a}, \quad z_2^{2a} = y^{1a}, \quad z_2^{3a} = 3y^{2a} + y^{1b}M_{2b}^{3a} \\ &z_3^{1a} = y^{1a}, \quad z_3^{2a} = 2y^{2a} + y^{1b}M_{1b}^{2a} \\ &z_3^{3a} = 3y^{3a} + 2y^{2b}M_{2b}^{3a} + y^{1b}M_{1b}^{3a}. \end{aligned} \tag{2.20}$$

The proof is obtained by (1.18). All z from (1.20) with respect to (1.3) are transformed as tensors of type (1,0).

Theorem 2.7 *The J structure transforms the Liouville vector fields in the following way:*

$$J\Gamma_{(1)} = 0 \quad J\Gamma_{(2)} = 3\Gamma_{(1)} \quad J\Gamma_{(3)} = 2\Gamma_{(2)}. \tag{2.21}$$

The proof follows from (2.16) and (2.18).

Theorem 2.8 *The Liouville vector fields in the basis B' have the coordinates*

$$\begin{aligned}\Gamma_{(1)} &= z_1'^{3a}\delta'_{3a} \quad \Gamma_{(2)} = z_2'^{2a}\delta'_{2a} + z_2'^{3a}\delta'_{3a}\\ \Gamma_{(3)} &= z_3'^{1a}\delta'_{1a} + z_3'^{2a}\delta'_{2a} + z_3'^{3a}\delta'_{3a},\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}6z_1'^{3a} &= 2z_2'^{2a} = z_3'^{1a} = y^{1a}\\ 2z_2'^{3a} &= z_3'^{2a} = y^{2a} + M_{0b}^{1a}y^{1b}\\ z_3'^{3a} &= \frac{1}{2}y^{3a} + M_{0b}^{1a}y^{2b} + M_{0b}^{2a}y^{1b}.\end{aligned} \tag{2.23}$$

Proof. If we substitute ∂_{1a}, ∂_{2a} and ∂_{3a} from (2.8) into (2.18) and compare with (2.19) we obtain (2.23).

Remark. $\Gamma_{(1)}$, $\Gamma_{(2)}$ and $\Gamma_{(3)}$ determined by (2.19) and (2.22) satisfy (2.21).

3 The 3-sprays

Definition 3.1 *A* 3*-spray on E is a vector field $S \in \chi(E)$, with the property*

$$JS = \Gamma_{(3)}, \tag{3.1}$$

where $\Gamma_{(3)}$ (see (2.1)) has the form

$$\Gamma_{(3)} = y^{1a}\partial_{1a} + 2y^{2a}\partial_{2a} + 3y^{3a}\partial_{3a}. \tag{3.2}$$

Remark. As $(3-(i-1))!\Gamma_{(i)} = \Gamma^{(i)}$ ($\Gamma^{(i)}$ is the notation of Liouville vector field used by R.Miron), so we have $\Gamma_{(3)} = \Gamma^{(3)}$.

Definition 3.2 *A curve $c : I \to M$ is a* 3*-path on M if its* 3*-extension $\tilde{c}$ is the integral curve of a* 3*-spray.*

Let us denote the curve $\tilde{c}$ in $E = Osc^3M$ by

$$\tilde{c}(t) : (x^i(t), y^{1i}(t), y^{2i}(t), y^{3i}(t)). \tag{3.3}$$

The position vector $r(t)$ of arbitrary point on $\tilde{c}(t)$ is given by

$$r(t) = y^{0i}\partial_{0i} + y^{1i}\partial_{1i} + y^{2i}\partial_{2i} + y^{3i}\partial_{3i}. \tag{3.4}$$

If we introduce the notation

$$\Gamma(t) = y^{0i}\partial_{0i} + y^{1i}\partial_{1i} + y^{2i}\partial_{2i}, \tag{3.5}$$

then (3.4) can be written in the form

$$r(t) = \Gamma(t) + y^{3i}\partial_{3i}. \tag{3.6}$$

The tangent vector of $\tilde{c}(t)$ is $dr(t) = \dot{r}(t)dt$ and in the basis $\bar{B}$ it can be expressed as

$$dr = dy^{0i}\partial_{0i} + dy^{1i}\partial_{1i} + dy^{2i}\partial_{2i} + dy^{3i}\partial_{3i}. \tag{3.7}$$

Theorem 3.1 *The tangent vector dr of $\tilde{c}(t)$ in the basis B' has the form*

$$dr = \delta' y^{0i}\delta'_{0i} + \delta' y^{1i}\delta'_{1i} + \delta' y^{2i}\delta'_{2i} + \delta' y^{3i}\delta'_{3i}. \tag{3.8}$$

Proof. From (2.8) we have

$$\begin{aligned}
6\partial_{3a} &= \delta'_{3a} \\
2\partial_{2a} &= \delta'_{2a} + N^{1b}_{0a}\delta'_{3b} \\
\partial_{1a} &= \delta'_{1a} + N^{1b}_{0a}(\delta'_{2b} + N^{1c}_{0b}\delta'_{3c}) + N^{2b}_{0a}\delta'_{3b} \\
\partial_{0a} &= \delta'_{0a} + N^{1b}_{0a}[\delta'_{1b} + N^{1c}_{0b}(\delta'_{2c} + N^{1d}_{0c}\delta'_{3d}) + N^{2c}_{0b}\delta'_{3c} + \\
&\quad N^{2b}_{0a}(\delta'_{2b} + N^{1c}_{0b}\delta'_{3c}) + N^{3b}_{0a}\delta'_{3b}.
\end{aligned}$$

The substitution of the above equations into (3.7) results

$$\begin{aligned}
\delta r &= dy^{0a}\delta'_{0a} + (dy^{1a} + N^{1a}_{0b}dy^{0b})\delta'_{1a} + \\
&\quad [\frac{1}{2}dy^{2a} + N^{1a}_{0b}dy^{1b} + (N^{2a}_{0c} + N^{1a}_{0d}N^{1d}_{0c})dy^{0c}]\delta'_{2a} + \\
&\quad [\frac{1}{6}dy^{3a} + \frac{1}{2}N^{1a}_{0b}dy^{2b} + (N^{2a}_{0b} + N^{1c}_{0b}N^{1a}_{0c})dy^{1b} + \\
&\quad (N^{3a}_{0b} + N^{2a}_{0c}N^{1c}_{0b} + N^{1a}_{0c}N^{2c}_{0b} + N^{1c}_{0a}N^{1d}_{0c}N^{1a}_{1d})dy^{0b}]\delta'_{3a}.
\end{aligned}$$

Using (2.9a) and (2.5) the above equation takes the form (3.8).

From (3.8) it is clear, that dr is a vector field with respect to the coordinate transformations of form (1.1), i.e.

$$dr = \delta' y^{0i'}\delta'_{0i'} + \delta' y^{1i'}\delta'_{1i'} + \delta' y^{2i'}\delta'_{2i'} + \delta' y^{3i'}\delta'_{3i'}.$$

Using the notations $\dot{r} = \frac{dr}{dt}$, $\dot{\Gamma} = \frac{d\Gamma}{dt}$, from (3.6) we obtain

$$\dot{r} = \dot{\Gamma} + \frac{dy^{3i}}{dt}\partial_{3i}, \tag{3.9}$$

where

$$\dot{\Gamma} = y^{1i}\partial_{0i} + y^{2i}\partial_{1i} + y^{3i}\partial_{2i}. \tag{3.10}$$

Proposition 3.1 . $\dot{\Gamma}$ *is not a vector field, but it has the property*

$$J\dot{\Gamma} = \Gamma_{(3)}. \tag{3.11}$$

Proof. If we apply the linear transformation J on $\dot{\Gamma}$ and use (2.16) and (3.2), we get

$$\begin{aligned} J\dot{\Gamma} &= y^{1i}J\partial_{0i} + y^{2i}J\partial_{1i} + y^{3i}J\partial_{2i} = \\ & y^{1i}\partial_{1i} + 2y^{2i}\partial_{2i} + 3y^{3i}\partial_{3i} = \Gamma_{(3)}. \end{aligned}$$

Theorem 3.2 *The vector fields S, which has the property $JS = \Gamma_{(3)}$ can be written in the form*

$$S = \dot{\Gamma} + G^{3i}\partial_{3i}, \tag{3.12}$$

where G^{3i} are such functions, which under (1.1) transform in the following way:

$$G^{3i'} = G^{3i}\partial_{3i}y^{3i'} + \dot{\Gamma}(y^{3i'}). \tag{3.13}$$

Proof. From (3.12), (3.11) and (2.16) it is obvious, that

$$JS = J\dot{\Gamma} + G^{3i}J\partial_{3i} = J\dot{\Gamma} = \Gamma_{(3)}.$$

From the above equation it can be seen that $JS = \Gamma_{(3)}$ is satisfied, when in (3.12) G^{3i} are arbitrary functions. When S is a vector field, G^{3i} must satisfy some special conditions with respect to (1.1). S is a vector field if

$$\begin{aligned} S &= y^{1i}\partial_{0i} + y^{2i}\partial_{1i} + y^{3i}\partial_{2i} + G^{3i}\partial_{3i} = \\ & y^{1i'}\partial_{0i'} + y^{2i'}\partial_{1i'} + y^{3i'}\partial_{2i'} + G^{3i'}\partial_{3i'}. \end{aligned} \tag{3.14}$$

Substituting $\partial_{0i}, \partial_{1i}, \partial_{2i}, \partial_{3i}$ from (1.6) into (3.14) and equating the coefficients beside the basis vectors, we get

$$\begin{aligned} &\partial_{0i'} : y^{1i'} = y^{1i}\partial_{0i}y^{0i'} \\ &\partial_{1i'} : y^{2i'} = y^{1i}\partial_{0i}y^{1i'} + y^{2i'}\partial_{1i}y^{1i'} \\ &\partial_{2i'} : y^{3i'} = y^{1i}\partial_{0i}y^{2i'} + y^{2i}\partial_{1i}y^{2i'} + y^{3i}\partial_{2i}y^{2i'} \\ &\partial_{3i'} : G^{3i'} = G^{3i}\partial_{3i}y^{3i'} + y^{1i}\partial_{0i}y^{3i'} + y^{2i}\partial_{1i}y^{3i'} + y^{3i}\partial_{2i}y^{3i'} = \\ &\qquad G^{3i}\partial_{3i}y^{3i'} + \dot{\Gamma}(y^{3i'}). \end{aligned} \tag{3.15}$$

The first 3 equations in (3.15) are exactly the allowable coordinate transformatins given by (1.1).

Theorem 3.3 *The vector field S is the tangent vector of the curve $\tilde{c}(t)$ given by (3.3) if and only if the functions G^{3i} beside transformation law (3.13) satisfy the relations*

$$G^{3i} = \frac{dy^{3i}}{dt}. \tag{3.16}$$

Proof. From (3.9) and (3.12):

$$\dot{r} = \dot{\Gamma} + \frac{dy^{3i}}{dt}\partial_{3i}, \quad S = \dot{\Gamma} + G^{3i}\partial_{3i}$$

it is obvious that S is parallel to $\dot{r}$ if and only if (3.16) is satisfied.

The transformation law of G^{3i} can be expressed in function of M's.

Theorem 3.4 *When S is a spray with spray coefficients G^{3i}, then S can be written in the form*

$$S = S^{0i}\delta_{0i} + S^{1i}\delta_{1i} + S^{2i}\delta_{2i} + S^{3i}\delta_{3i}, \tag{3.17}$$

where

$$S^{0i} = y^{1i} \tag{3.18}$$

$$S^{1i} = y^{2i} + y^{1j}M^{1i}_{0j}$$

$$S^{2i} = y^{3i} + y^{2j}M^{2i}_{1j} + y^{1j}M^{2i}_{0j}$$

$$S^{3i} = G^{3i} + y^{3j}M^{3i}_{2j} + y^{2j}M^{3i}_{1j} + y^{1j}M^{3i}_{0j}.$$

$S^{\alpha i}$ $(\alpha = 0, 1, \ldots, 3)$ *are d-tensors of type (1,0) i.e.*

$$S^{\alpha i'} = \frac{\partial x^{i'}}{\partial x^i}S^{\alpha i}. \tag{3.19}$$

For $\alpha = 3$ we have

$$G^{3i'} + y^{3j'}M^{3i'}_{2j'} + y^{2j'}M^{2i'}_{1j'} + y^{1j'}M^{2i'}_{0j'} = \tag{3.20}$$

$$\frac{\partial x^{i'}}{\partial x^i}(G^{3i} + y^{3j}M^{3i}_{2j} + y^{2j}M^{3i}_{1j} + y^{1j}M^{3i}_{0j})$$

Proof. Substituting $\partial_{0i}, \partial_{1i}, \partial_{2i}, \partial_{3i}$ from (1.18) into the first equation of (3.14) we obtain (3.17) and (3.18). From (3.17) and (3.18) follow (3.19) and (3.20).

Theorem 3.5 *The vector fields* $\Gamma_{(1)}, \ldots, \Gamma_{(3)}$*,* S *and the linear transformation* J *are connected by*

$$\begin{aligned} JS &= \Gamma_{(3)} \\ J^2S &= 2!\Gamma_{(2)} \\ J^3S &= 3!\Gamma_{(1)} \\ J^4S &= 0. \end{aligned} \tag{3.21}$$

Proof. (3.21) follows from (2.21) and (3.1).

4 Zermello's conditions in Osc^3M

The integral of action I_{c^*} does not depend on the parametrization of the curve c^* if

$$\int_0^1 L(x, y^1, y^2, y^3)dt = \int_0^1 L(x, y^{1'}, y^{2'}, y^{3'})ds, \tag{4.1}$$

for any change of parameter $s = s(t)$, where $s(t)$ is at least C^4 function, $s'(t) > 0$, $s(0) = 0$, $s(1) = 1$, and

$$y^{\alpha a'} = d_s^\alpha x^a = \frac{d^\alpha x^a}{ds^\alpha}, \quad \alpha = 1, 2, 3.$$

(4.1) will be satisfied if

$$L(x, y^1, y^2, y^3) = L(x, y^{1'}, y^{2'}, y^{3'})s', \tag{4.2}$$

where $s' = \frac{ds}{dt}$. We shall use the notation

$$s^{(\alpha)} = \frac{d^\alpha s}{dt^\alpha}, \quad \alpha = 1, 2, 3.$$

The equations which give the invariance of I_{c^*} from the parametrization of the curve c^* are called Zermello's conditions.By pure calculation we get:

$$\begin{aligned} y^{1a} &= y^{1a'}s', \\ y^{2a} &= y^{2a'}(s')^2 + y^{1a'}s'', \\ y^{3a} &= y^{3a'}(s')^3 + y^{2a'}3s's'' + y^{1a'}s''', \\ \frac{dy^{3a}}{dt} &= \frac{dy^{3a'}}{ds}(s')^4 + y^{3a'}6s'^2s'' + y^{2a'}(3(s'')^2 + 4s's''') + y^{1a'}s'^v. \end{aligned} \tag{4.3}$$

Taking the partial derivatives of (4.2) with respect to s', s'', s''' and s'^v we get:

$$(\partial_{1a}L)y^{1a'} + (\partial_{2a}L)2s'y^{2a'} + (\partial_{3a}L)(3(s')^2y^{3a'} + 3s''y^{2a'}) = L(x, y^{1'}, y^{2'}, y^{3'}), \tag{4.4}$$

$$(\partial_{2a}L)y^{1a'} + (\partial_{3a}L)(3s'y^{2a'}) = 0 \tag{4.5}$$

$$(\partial_{3a}L)y^{1a'} = 0. \tag{4.6}$$

In (4.4)-(4.6) $L = L(x, y^1, y^2, y^3)$. If we multiply (4.4) with s', (4.5) with $2s''$, (4.6) with $3s'''$ and add all these equations we obtain:

$$(\partial_{1a}L)y^{1a'}s' + 2(\partial_{2a}L)(y^{2a'}(s')^2 + y^{1a'}s'') +$$
$$3(\partial_{3a}L)(y^{3a'}(s')^3 + 3y^{2a'}s's'' + y^{1a'}s''') = L(x, y^{1'}, y^{2'}, y^{3'})s'.$$

The substitution of (4.3) and (4.2) into above equations results

$$(\partial_{1a}L)y^{1a} + 2(\partial_{2a}L)y^{2a} + 3(\partial_{3a}L)y^{3a} = L. \tag{4.7}$$

If we (4.5) multiply with s', (4.6) with $3s''$ and add all such obtained equations we get:

$$(\partial_{2a}L)(y^{1a'}s') + 3(\partial_{3a}L)(y^{2a'}(s')^2 + y^{1a'}s'') = 0,$$

i.e.

$$(\partial_{2a}L)y^{1a} + 3(\partial_{3a}L)y^{2a} = 0. \tag{4.8}$$

From (4.6) it follows:

$$(\partial_{3a}L)y^{1a} = 0. \tag{4.9}$$

Theorem 4.1 *Equation (4.7)-(4.9) are the Zermello's conditions in* Osc^3M.

The comparation of (4.7)-(4.9) with (1.21) results.

Theorem 4.2 *The Zermello's condition in* Osc^3M *are:*

$$\Gamma_{(1)}L = 0, \quad \Gamma_{(2)}L = 0, \quad \Gamma_{(3)}L = L.$$

They are the necessary conditions for the invariance of I_{c^*} from the parametrization of the curve c^*.

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