Kragujevac J. Math. 25 (2003) 219–225.

SOME TRIGONOMETRIC RELATIONS IN THE LORENTZIAN PLANE

Emilija Nešović and Miroslava Petrović–Torgašev

Faculty of Science, P. O. Box 60, 34000 Kragujevac, Serbia and Montenegro

Abstract. In this paper, by using the definition of oriented hyperbolic angle between two non–null vectors in the Lorentzian plane L^2, we characterize the area of a triangle and obtain some Lorentzian trigonometric relations. In particular, we study the hyperbolic sine law and the hyperbolic cosine law holding in a triangle.

1. Introduction

One of the basic notions in the Lorentzian plane geometry is the notion of so–called hyperbolic angle between two vectors (directions). The notion of oriented hyperbolic angle between any two timelike vectors is defined in [1,2], where the authors studied the main properties of the hyperbolic angle function. Some of the mentioned properties can be also found in [5]. The notion of oriented hyperbolic angle is further extended in [4]. Moreover, in [4] the authors have defined oriented hyperbolic angle between any two spacelike vectors as well as between a spacelike vector and a timelike vector. They also defined a measure on the set of oriented hyperbolic angles.

In this paper, using definitions of oriented hyperbolic angles between non–null vectors, we obtain some Lorentzian trigonometric relations holding in a triangle. In particular, we study the hyperbolic sine law and the hyperbolic cosine law.

2. Preliminaries

The Lorentzian space L^n is the vector space R^n provided with the Lorentzian inner product $< \cdot, \cdot >$ given by

$$< X, Y >= x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n$$

where $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$. Since $< \cdot, \cdot >$ is indefinite metric, a vector $V \in L^n$ can have one of three causal characters: it can be spacelike if $< V, V > 0$ or $V = 0$, timelike if $< V, V >< 0$ and null (lightlike) if $< V, V >= 0$

and $V \neq 0$. The norm of a vector v is given by $||V|| = \sqrt{| < V, V > |}$ and two vectors $V, W \in L^n$ are said to be orthogonal if $< V, W >= 0$. In particular, in the Lorentzian plane L^2, let $E = (0, 1)$. Then the time–orientation is defined in the following way. A non–null vector $V = (v_1, v_2)$ is said to be respectively future–pointing or past–pointing, if $< V, E >< 0$ or $< V, E >> 0$.

In the sequel, we recall the definitions of the hyperbolic angle between two non–null vectors. Denote by G the proper Lorentz group consisting of all matrices of the form

$$A(u) = \begin{bmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{bmatrix}, \qquad u \in R.$$

Then G is the group of all linear transformations of L^2 which preserve inner product, orientation and time–orientation.

Recall that if V and W are both unit future–pointing (past–pointing) timelike vectors in L^2, then the oriented hyperbolic angle from V to W is defined in [1] to be the number $u \in R$ such that $A(u)V = W$ (a.1). In particular, the oriented hyperbolic angle from a unit future–pointing timelike vector V to a unit past–pointing timelike vector $-W$ is defined in [2] to be the number $-u \in R$ such that $-A(-u)(V) = -W$ (a.2).

Further, denote by D the matrix of the Euclidean reflection D in the first diagonal $\{(x, x)|x \in R\}$ of R^2, given by

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we have $DA(u) = A(u)D$. Let $B(u) = A(u)D$. Then the matrix $B(u)$ is given by:

$$B(u) = \begin{bmatrix} \sinh u & \cosh u \\ \cosh u & \sinh u \end{bmatrix}, \qquad u \in R.$$

If $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ are two unit spacelike vectors with sgn $x_1 =$ sgn y_1, then $u \in R$ is defined in [4] to be the oriented hyperbolic angle from X to Y if $A(u)X = Y$ (b.1). Moreover, if sgn $x_1 \neq$ sgn y_1, then $u \in R$ is said to be the oriented hyperbolic angle from X to Y, if $-A(u)X = Y$ (b.2).

Finally, recall that the oriented hyperbolic angle between a unit spacelike vector $X = (x_1, x_2)$ and a unit timelike vector $V = (v_1, v_2)$ is defined in [4] as follows. If sgn $x_1 =$ sgn v_2, then $u \in R$ is said to be the oriented hyperbolic angle from X to V if $B(u)X = V$ (c.1). On the other hand, if sgn $x_1 \neq$ sgn v_2, then $u \in R$ is said to be the oriented hyperbolic angle from X to V, if $-B(u)X = V$ (c.2).

Note that the above definitions for the oriented angle $(X, Y) = u$ between unit vectors X and Y amount to the following:

(a.1) $\cosh(u) = - < X, Y >, \quad \sinh(u) = - < X, DY >$;
(a.2) $\cosh(u) =< X, Y >, \quad \sinh(u) =< X, DY >$;
(b.1) $\cosh(u) =< X, Y > \quad \sinh(u) =< X, DY >$;
(b.2) $\cosh(u) = - < X, Y > \quad \sinh(u) = - < X, DY >$;
(c.1) $\cosh(u) =< X, DY > \quad \sinh(u) =< X, Y >$;

(c.2) $\cosh(u) = -<X, DY>$ $\quad \sinh(u) = -<X, Y>$.

Therefore, for vectors X and Y of arbitrary lengths $||X|| \neq 0 \neq ||Y||$, for instance in case (b.1) we have:

$$\text{(b.1')} \qquad \cosh(u) = \frac{<X, Y>}{||X||\,||Y||}, \quad \sinh(u) = \frac{<X, DY>}{||X||\,||Y||}.$$

3. Some trigonometric relations in the Lorentzian plane $\mathbf{L}^2$

In this section, using the notion of hyperbolic angle between two non–null vectors, we characterize the area of a triangle in the Lorentzian plane and give some Lorentzian trigonometric relations. Recall that in the Lorentzian plane the area of the parallelogram spanned by vectors $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ is given in [1] by $\mathcal{A} = |x_1 y_2 - x_2 y_1| = |<X, DY>|$. It follows that the area S of the triangle $\triangle ABC$ is equal to one half of the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, i.e. it is given by the formula $S = |<\overrightarrow{AB}, D(\overrightarrow{AC})>|/2$. In the following theorems, $(\overrightarrow{X}, \overrightarrow{Y})$ will denote the oriented hyperbolic angle from the vector $\overrightarrow{X}$ to the vector $\overrightarrow{Y}$.

Theorem 3.1. *If $\overrightarrow{AB} = (x_1, x_2)$, $\overrightarrow{AC} = (y_1, y_2)$ are two noncollinear spacelike vectors, then the area of the triangle ABC is given by*

$$S = \frac{||\overrightarrow{AB}||\,||\overrightarrow{AC}||\,|\sinh(\overrightarrow{AB}, \overrightarrow{AC})|}{2}.$$

Proof. Since $S = |<\overrightarrow{AB}, D(\overrightarrow{AC})>|/2$, using the equations (b.1') and (b.2'), we easily get the above equation. $\square$

It is proved in [1] that theorem 3.1 is also valid when vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are timelike.

Theorem 3.2. *If $\overrightarrow{AB} = (x_1, x_2)$, $\overrightarrow{AC} = (y_1, y_2)$ are a spacelike and a timelike vectors respectively, then the area of the triangle ABC is given by*

$$S = \frac{||\overrightarrow{AB}||\,||\overrightarrow{AC}||\,\cosh(\overrightarrow{AB}, \overrightarrow{AC})}{2}.$$

Proof. Using formula for the area S and equations (c.1') and (c.2'), we easily obtain the above equation. $\square$

Theorem 3.3. *If $\overrightarrow{AB} = (x_1, x_2)$ and $\overrightarrow{AC} = (z_1, z_2)$ are two noncollinear spacelike vectors and $\overrightarrow{BC} = (y_1, y_2)$ is timelike vector such that $g(\overrightarrow{AB}, \overrightarrow{BC}) = 0$, then*

$$\cosh(\overrightarrow{AB}, \overrightarrow{AC}) = \cosh(\overrightarrow{BC}, \overrightarrow{AC}) = ||\overrightarrow{AB}||/||\overrightarrow{AC}||,$$

$$|\sinh(\overrightarrow{AB}, \overrightarrow{AC})| = |\sinh(\overrightarrow{BC}, \overrightarrow{AC})| = ||\overrightarrow{BC}||/||\overrightarrow{AC}||.$$

Proof. Since $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$, it follows that

$$x_1+y_1=z_1,\quad x_2+y_2=z_2. \tag{3.1}$$

First we prove that sgn $x_1=$ sgn z_1. If sgn $x_1\neq$ sgn z_1, then $0<<\overrightarrow{AB},\overrightarrow{AC}>^2<2x_1z_1<\overrightarrow{AB},\overrightarrow{AC}>$ and thus $<\overrightarrow{AB},\overrightarrow{AC}><0$. On the other hand, $<\overrightarrow{AB},\overrightarrow{AC}>=||\overrightarrow{AB}||^2>0$, which is a contradiction. Therefore, sgn $x_1=$ sgn z_1 and we distinguish two cases: (1°) sgn $x_1=$ sgn $z_1=$ sgn y_2; (2°) sgn $x_1=$ sgn $z_1\neq$ sgn y_2.

(1°). Then $\cosh(\overrightarrow{AB},\overrightarrow{AC})=<\overrightarrow{AB},\overrightarrow{AC}>/||\overrightarrow{AB}||\,||\overrightarrow{AC}||=||\overrightarrow{AB}||/||\overrightarrow{AC}||$, $\sinh(\overrightarrow{AB},\overrightarrow{AC})=<\overrightarrow{AB},D(\overrightarrow{AC})>/||\overrightarrow{AB}||\,||\overrightarrow{AC}||$, which together with (3.1) gives

$$\sinh(\overrightarrow{AB},\overrightarrow{AC})=<\overrightarrow{AB},D(\overrightarrow{BC})>/||\overrightarrow{AB}||\,||\overrightarrow{AC}||. \tag{3.2}$$

Since $\sinh(\overrightarrow{AB},\overrightarrow{BC})=<\overrightarrow{AB},\overrightarrow{BC}>/||\overrightarrow{AB}||\,||\overrightarrow{BC}||=0$, it follows that $(\overrightarrow{AB},\overrightarrow{BC})=0$. Consequently, $\cosh(\overrightarrow{AB},\overrightarrow{BC})=<\overrightarrow{AB},D(\overrightarrow{BC})>/||\overrightarrow{AB}||\,||\overrightarrow{BC}||=1$. It follows that

$$<\overrightarrow{AB},D(\overrightarrow{BC})>=||\overrightarrow{AB}||\,||\overrightarrow{BC}||. \tag{3.3}$$

Substituting (3.3) into (3.2) we obtain $\sinh(\overrightarrow{AB},\overrightarrow{AC})=||\overrightarrow{BC}||/||\overrightarrow{AC}||$. Besides, $\sinh(\overrightarrow{BC},\overrightarrow{AC}=-<\overrightarrow{BC},\overrightarrow{AC}>/||\overrightarrow{BC}||\,||\overrightarrow{AC}||=||\overrightarrow{BC}||/||\overrightarrow{AC}||$, $\cosh(\overrightarrow{BC},\overrightarrow{AC})=<\overrightarrow{AC},D(\overrightarrow{BC})>/||\overrightarrow{BC}||\,||\overrightarrow{AC}||$, which together with (3.1) gives $\cosh(\overrightarrow{BC},\overrightarrow{AC})=<\overrightarrow{AB},D(\overrightarrow{BC})>/||\overrightarrow{BC}||\,||\overrightarrow{AC}||$. Substituting (3.3) into the last equation, we find $\cosh(\overrightarrow{BC},\overrightarrow{AC})=||\overrightarrow{AB}||/||\overrightarrow{AC}||$.

(2°). In this case we have $\cosh(\overrightarrow{AB},\overrightarrow{AC})=||\overrightarrow{AB}||/||\overrightarrow{AC}||$, $\sinh(\overrightarrow{AB},\overrightarrow{AC})=<\overrightarrow{AB},D(\overrightarrow{AC})>/||\overrightarrow{AB}||\,||\overrightarrow{AC}||$ which together with (3.1) implies that

$$\sinh(\overrightarrow{AB},\overrightarrow{AC})=<\overrightarrow{AB},D(\overrightarrow{BC})>/||\overrightarrow{AB}||\,||\overrightarrow{AC}||. \tag{3.4}$$

Since $\sinh(\overrightarrow{AB},\overrightarrow{BC})=-<\overrightarrow{AB},\overrightarrow{BC}>/||\overrightarrow{AB}||\,||\overrightarrow{BC}||=0$, we find $(\overrightarrow{AB},\overrightarrow{BC})=0$. Therefore, $\cosh(\overrightarrow{AB},\overrightarrow{BC})=-<\overrightarrow{AB},D(\overrightarrow{BC})>/||\overrightarrow{AB}||\,||\overrightarrow{BC}||=1$ and thus

$$-<\overrightarrow{AB},D(\overrightarrow{BC})>=||\overrightarrow{AB}||\,||\overrightarrow{BC}||. \tag{3.5}$$

Substituting (3.5) into (3.4), we find $\sinh(\overrightarrow{AB},\overrightarrow{AC})=-||\overrightarrow{BC}||/||\overrightarrow{AC}||$. Next, $\sinh(\overrightarrow{BC},\overrightarrow{AC})=<\overrightarrow{BC},\overrightarrow{AC}>/||\overrightarrow{BC}||\,||\overrightarrow{AC}||=-||\overrightarrow{BC}||/||\overrightarrow{AC}||$, $\cosh(\overrightarrow{BC},\overrightarrow{AC})=-<\overrightarrow{AC},D(\overrightarrow{BC})>/||\overrightarrow{BC}||\,||\overrightarrow{AC}||$ which together with (3.1) gives $\cosh(\overrightarrow{BC},\overrightarrow{AC})=-<\overrightarrow{AB},D(\overrightarrow{BC})>/||\overrightarrow{BC}||\,||\overrightarrow{AC}||$. Substituting (3.5) into the last equation, we obtain $\cosh(\overrightarrow{BC},\overrightarrow{AC})=||\overrightarrow{AB}||/||\overrightarrow{AC}||$. □

Theorem 3.4. *If $\overrightarrow{BC}=(y_1,y_2)$ and $\overrightarrow{AC}=(z_1,z_2)$ are two noncollinear timelike vectors and $\overrightarrow{AB}=(x_1,x_2)$ is spacelike vector such that $g(\overrightarrow{AB},\overrightarrow{BC})=0$, then*

$$\cosh(\overrightarrow{AB},\overrightarrow{AC})=\cosh(\overrightarrow{BC},\overrightarrow{AC})=||\overrightarrow{BC}||/||\overrightarrow{AC}||,$$
$$|\sinh(\overrightarrow{AB},\overrightarrow{AC}|=|\sinh(\overrightarrow{BC},\overrightarrow{AC})|=||\overrightarrow{AB}||/||\overrightarrow{AC}||.$$

Proof. The proof is similar to the proof of the theorem 3.3. First we shall prove that sgn y_2 = sgn z_2. If sgn $y_2 \neq$ sgn z_2, then $0 << \overrightarrow{BC}, \overrightarrow{AC} >^2 < -2y_2 z_1 g(\overrightarrow{BC}, \overrightarrow{AC})$, so it follows that $< \overrightarrow{BC}, \overrightarrow{AC} >> 0$. On the other hand, $< \overrightarrow{BC}, \overrightarrow{AC} >=< \overrightarrow{BC}, \overrightarrow{BC} >= -||\overrightarrow{BC}||^2 < 0$, which is a contradiction. Consequently, sgn y_2 = sgn z_2 and we distinguish two cases: (1°) sgn x_1 = sgn y_2 = sgn z_2; (2°) sgn $x_1 \neq$ sgn y_2 = sgn z_2.

(1°). Then $\sinh(\overrightarrow{AB}, \overrightarrow{AC}) = ||\overrightarrow{AB}||/||\overrightarrow{AC}||$, $\cosh(\overrightarrow{AB}, \overrightarrow{AC}) =< \overrightarrow{AB}, D(\overrightarrow{BC}) > /||\overrightarrow{AB}||\,||\overrightarrow{AC}|| = ||\overrightarrow{BC}||/||\overrightarrow{AC}||$. It follows that $\cosh(\overrightarrow{BC}, \overrightarrow{AC}) = ||\overrightarrow{BC}||/||\overrightarrow{AC}||$, $\sinh(\overrightarrow{BC}, \overrightarrow{AC}) =< \overrightarrow{AB}, D(\overrightarrow{BC}) > /||\overrightarrow{BC}||\,||\overrightarrow{AC}|| = ||\overrightarrow{AB}||/||\overrightarrow{AC}||$.

(2°). In this case $\sinh(\overrightarrow{AB}, \overrightarrow{AC}) = -||\overrightarrow{AB}||/||\overrightarrow{AC}||$, $\cosh(\overrightarrow{AB}, \overrightarrow{AC}) = ||\overrightarrow{BC}||/||\overrightarrow{AC}||$. Moreover, $\cosh(\overrightarrow{BC}, \overrightarrow{AC}) = ||\overrightarrow{BC}||/||\overrightarrow{AC}||$, $\sinh(\overrightarrow{BC}, \overrightarrow{AC}) = -||\overrightarrow{AB}||/||\overrightarrow{AC}||$. □

Remark 3.1. The equations in theorems 3.3 and 3.4 also hold if the words "spacelike" and "timelike" are reversed.

In the following theorems, we study the hyperbolic sine law and the hyperbolic cosine law which hold in a triangle. These laws were studied in [1] for a triangle ABC such that $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are all future–pointing timelike vectors.

Theorem 3.5 (Hyperbolic sine law). *If $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are three non-collinear spacelike vectors and $\alpha = (\overrightarrow{AB}, \overrightarrow{AC})$, $\beta = (\overrightarrow{AB}, \overrightarrow{BC})$, $\gamma = (\overrightarrow{AC}, \overrightarrow{BC})$, then in the triangle ABC there holds:*

$$\frac{||\overrightarrow{AC}||}{|\sinh\beta|} = \frac{||\overrightarrow{BC}||}{|\sinh\alpha|} = \frac{||\overrightarrow{AB}||}{|\sinh\gamma|}. \tag{3.6}$$

Proof. Let E be a point on the line AB such that $\overrightarrow{CE}$ is timelike and $< \overrightarrow{CE}, \overrightarrow{AB} >= 0$. By the theorem 3.3 in triangle AEC there holds $|\sinh\alpha| = ||\overrightarrow{CE}||/||\overrightarrow{AC}||$. Next, by the theorem 3.4 in triangle BEC there holds $|\sinh\beta| = ||\overrightarrow{CE}||/||\overrightarrow{BC}||$. Therefore,

$$||\overrightarrow{BC}||\,|\sinh\beta| = ||\overrightarrow{AC}||\,|\sinh\alpha|. \tag{3.7}$$

Further, let F be a point on the line AC such that $\overrightarrow{BF}$ is timelike and $< \overrightarrow{BF}, \overrightarrow{AC} > = 0$. Then by the theorem 3.3 in the triangle ABF there holds $|\sinh\alpha| = ||\overrightarrow{BF}||/||\overrightarrow{AB}||$. Also, by the theorem 3.4 in triangle BFC holds $|\sinh\gamma| = ||\overrightarrow{BF}||/||\overrightarrow{BC}||$ and thus

$$|\sinh\alpha|\,||\overrightarrow{AB}|| = |\sinh\gamma|\,||\overrightarrow{BC}||. \tag{3.8}$$

Finally, equations (3.7) and (3.8) imply equation (3.6). □

Theorem 3.6 (Hyperbolic sine law). *If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are two noncollinear spacelike (timelike) vectors, $\overrightarrow{BC}$ is the timelike (spacelike) vector and $\alpha = (\overrightarrow{AB}, \overrightarrow{AC})$, $\beta = (\overrightarrow{AB}, \overrightarrow{BC})$, $\gamma = (\overrightarrow{AC}, \overrightarrow{BC})$, then in the triangle ABC there holds*

$$\frac{||\overrightarrow{AC}||}{\cosh\beta} = \frac{||\overrightarrow{BC}||}{|\sinh\alpha|} = \frac{||\overrightarrow{AB}||}{\cosh\gamma}. \tag{3.9}$$

Proof. We give the proof in the case when $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are spacelike vectors and $\overrightarrow{BC}$ is timelike vector. The proof in the case when $\overrightarrow{AB}$, $\overrightarrow{AC}$ are timelike and $\overrightarrow{BC}$ is spacelike is analogous. Let E be a point on the line AB such that $\overrightarrow{CE}$ is timelike and $<\overrightarrow{CE}, \overrightarrow{AB}>= 0$. Then by the theorem 3.3 in the triangle AEC there holds $|\sinh\alpha| = ||\overrightarrow{CE}||/||\overrightarrow{AC}||$. Next, by the theorem 3.4 in triangle BEC there holds $\cosh\beta = ||\overrightarrow{CE}||/||\overrightarrow{BC}||$. Consequently,

$$||\overrightarrow{AC}||\,|\sinh\alpha| = ||\overrightarrow{BC}||\cosh\beta. \tag{3.10}$$

Besides, let F be a point on the line AC such that $\overrightarrow{BF}$ is timelike and $<\overrightarrow{BF}, \overrightarrow{AC}> = 0$. Then by theorem 3.3 in triangle ABF we have $|\sinh\alpha| = ||\overrightarrow{BF}||/||\overrightarrow{AB}||$. Also, by theorem 3.4 in triangle BFC there holds $\cosh\gamma = ||\overrightarrow{BF}||/||\overrightarrow{BC}||$. Hence

$$||\overrightarrow{AB}||\,|\sinh\alpha| = ||\overrightarrow{BC}||\cosh\gamma. \tag{3.11}$$

Finally, equations (3.10) and (3.11) yield (3.9). □

Similarly, combining the causal characters of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$, we obtain the following two theorems.

Theorem 3.7 (Hyperbolic sine law). *If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are two noncollinear spacelike (timelike) vectors, $\overrightarrow{AC}$ is timelike (spacelike) vector and $\alpha = (\overrightarrow{AB}, \overrightarrow{AC})$, $\beta = (\overrightarrow{AB}, \overrightarrow{BC})$, $\gamma = (\overrightarrow{AC}, \overrightarrow{BC})$, then in the triangle ABC there holds:*

$$\frac{||\overrightarrow{AC}||}{|\sinh\beta|} = \frac{||\overrightarrow{BC}||}{\cosh\alpha} = \frac{||\overrightarrow{AB}||}{\cosh\gamma}.$$

Theorem 3.8 (Hyperbolic sine law). *If $\overrightarrow{BC}$ and $\overrightarrow{AC}$ are two noncollinear spacelike (timelike) vectors, $\overrightarrow{AB}$ is timelike (spacelike) vector and $\alpha = (\overrightarrow{AB}, \overrightarrow{AC})$, $\beta = (\overrightarrow{AB}, \overrightarrow{BC})$, $\gamma = (\overrightarrow{AC}, \overrightarrow{BC})$, then in the triangle ABC there holds:*

$$\frac{||\overrightarrow{AC}||}{\cosh\beta} = \frac{||\overrightarrow{BC}||}{\cosh\alpha} = \frac{||\overrightarrow{AB}||}{|\sinh\gamma|}.$$

Finally, in a similar way in the next two theorems, we obtain the hyperbolic cosine law.

Theorem 3.9 (Hyperbolic cosine law). *If* $\overrightarrow{AB}$, $\overrightarrow{BC}$ *and* $\overrightarrow{AC}$ *are three noncollinear spacelike vectors and* $\alpha = (\overrightarrow{AB}, \overrightarrow{AC})$, $\beta = (\overrightarrow{AB}, \overrightarrow{BC})$, $\gamma = (\overrightarrow{AC}, \overrightarrow{BC})$, *then in the triangle* ABC *there holds*

$$\begin{aligned} a^2 &= b^2 \mp 2bc\cosh\alpha + c^2, \\ b^2 &= a^2 \pm 2ac\cosh\beta + c^2, \\ c^2 &= a^2 \mp 2ab\cosh\gamma + b^2, \end{aligned}$$

where $||\overrightarrow{BC}|| = a$, $||\overrightarrow{AC}|| = b$, $||\overrightarrow{AB}|| = c$.

Proof. Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, using equations (b.1') and (b.2'), we obtain the above equations. □

Theorem 3.10 (Hyperbolic cosine law). *If* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$ *are two noncollinear spacelike vectors,* $\overrightarrow{BC}$ *is the timelike vector and* $\alpha = (\overrightarrow{AB}, \overrightarrow{AC})$, $\beta = (\overrightarrow{AB}, \overrightarrow{BC})$, $\gamma = (\overrightarrow{AC}, \overrightarrow{BC})$, *then in triangle* ABC *there holds*

$$\begin{aligned} a^2 &= -b^2 \pm 2bc\cosh\alpha - c^2, \\ b^2 &= c^2 \pm 2ac\sinh\beta - a^2, \\ c^2 &= b^2 \mp 2ab\sinh\gamma - a^2, \end{aligned}$$

where $||\overrightarrow{AB}|| = c$, $||\overrightarrow{BC}|| = a$, $||\overrightarrow{AC}|| = b$.

Proof. Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, using equations (b.1'), (b.2'), (c.1') and (c.2'), we get the above equations. □

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