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CHARACTERIZATIONS OF SPACELIKE GENERAL HELICES IN LORENTZIAN MANIFOLDS

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Dedicated to Professor Dr. Mileva Prvanović

Abstract. In this paper, some characterizations are given for a regular spacelike curve α to be a general helices in a Lorentzian Manifold $M_1^n (n \geq 3)$ by using its tangent vector field X, principal vector field Y and binormal vector field Z.

Particularly, these characterizations are investigated in the case of circular helix.

1 Introduction

T. Ikawa obtained in [4] the following equation

$$D_X^3 X - (k_1^2 - k_2^2) D_X X = 0$$

for the circular helices which corresponds to the case that k_1 and k_2 of a timelike curve α on the Lorentzian manifold $M_1^n (n \geq 3)$ are constants. Later in [1] N. Ekmekci and H. H. Hacısalihoğlu generalized T. Ikawa's result to the case of general helices and gave the following characterization,

$$D_X^3 X - \frac{3k_1^{'}}{k_1} D_X^2 X - \left\{ \frac{k_1^{''}}{k_1} - \frac{3(k_1^{'})^2}{k_1^2} + k_1^2 - k_2^2 \right\} D_X X = 0$$

for timelike curve with its tangent vector fields on any point. Later the similar characterization are obtained by using principal vector field and binormal vector field of the timelike curve in [2].

2 Preliminaries

Let M be an n-dimensional smooth manifold equipped with a metric g, where the metric g means a symmetric non-degenerate $(0, 2)$-tensor field on M with constant signature. A tangent space $T_P(M)$ at a point $P \in M$ is furnished with the canonical inner product. If the signature of the metric g is i, then we call M an *indefinite-Riemannian manifold* of signature i and denote by M_i. If g is positive definite, then M is a *Riemannian manifold.* Especially if $i = 1$, then M is called a *Lorentzian manifold.* A tangent vector x of M_i is said to be *spacelike*, if $g(x, x) > 0$ or $x = 0$, *timelike*, if $g(x, x) < 0$ and *null or lightlike* if $g(x, x) = 0$ and $x \neq 0$.

This terminology derives from the relativity theory. In a Lorentzian manifold M_1, timelike vectors and null vectors are called *causal vectors* [4],[6].

3 Curves

A curve in an indefinite-Riemannian manifold M_i is a smooth mapping $\alpha : I \to M_i$, where I is an open interval in the real line R^1. As an open submanifold of R^1, I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $t \in I$ is

$$\alpha^{'}(t) := dc(d/du \mid_t) \in T_{\alpha(t)}(M_i).$$

A curve $\alpha(t)$ is said to be regular if $\alpha^{'}(t)$ is not equal to zero for any t. If $\alpha(t)$ is a spacelike or timelike curve, we can reparameterize it such that $g(\alpha^{'}(t), \alpha^{'}(t)) = \epsilon$ (where $\epsilon = +1$ if α is spacelike and $\epsilon = -1$ if α is timelike, respectively). In this case $\alpha(t)$ is said to be unit speed or arc length parametrization [4],[6].

Here and in the sequel, we assume that the spacelike curve $\alpha(t)$ has an arc length parametrization.

Let $\alpha(t)$ be a spacelike curve in M_1. By $k_j(t)$, we denote the *j-th* curvature of $\alpha(t)$. If $k_j(t) \equiv 0$ for $j > 2$ and by X, we denote the tangent vector field, by Y, the principal vector field and by Z, the binormal vector field of $\alpha(t)$ spacelike curve, then we have the following Frenet formulas along $\alpha(t)$:

If Y is a timelike vector, then we have;

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{1}$$

where X, Y, Z are mutually orthogonal vectors satisfying equations

$$g(X,X) = g(Z,Z) = 1, g(Y,Y) = -1.$$

If Z is a timelike vector, then we have;

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{2}$$

where X,Y, and Z are mutually orthogonal vectors satisfying equations

$$g(X,X) = g(Y,Y) = 1, g(Z,Z) = -1.$$

4 Spacelike General Helices in Lorentzian Manifold

In a Lorentzian manifold M_1, a curve is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve is a *geodesic*. If only the first curvature is a non-zero constant and others are all identically zero, then the curve is called a *circle*. If the first and second curvatures are non-zero constants and others are all identically zero, then the curve is called a *helix* [1], [4].

If the first and second curvatures are not constant but $\dfrac{k_1}{k_2}$ is constant others are all identically zero, then the curve is a called a *general helix* [1].

In this section, we give some characterizations for a spacelike curve to be a general helix in Lorentzian Manifold.

Theorem 4.1. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a general helix if and only if*

$$D_X^3 X - \frac{3k_1'}{k_1} D_X^2 X - \left\{ \frac{k_1''}{k_1} - \frac{3(k_1')^2}{k_1^2} + k_1^2 + k_2^2 \right\} D_X X = 0 \tag{3}$$

or

$$D_X^3 X - \frac{3k_2'}{k_2} D_X^2 X - \left\{ \frac{k_1''}{k_1} - \frac{3(k_2')^2}{k_2^2} + k_1^2 + k_2^2 \right\} D_X X = 0 \tag{4}$$

Proof Suppose that $\alpha(t)$ is a general helix. Then from (1) it can easily be seen that given equations are satisfied. Conversely (3) (or (4)) holds. We show that the spacelike curve is a general helix. Then from (1), we have

$$Z = \frac{1}{k_2} D_X Y - \frac{k_1}{k_2} X \tag{5}$$

differentiating (5) and using (1), we obtain

$$D_X Z = \frac{1}{k_1 k_2} \left\{ D_X^3 X - \frac{3k_1'}{k_1} D_X^2 X - \left\{ \frac{k_1''}{k_1} - \frac{3(k_1')^2}{k_1^2} + k_1^2 + k_2^2 \right\} D_X X \right\} +$$
$$\frac{1}{k_1^2} \left(\frac{k_1}{k_2} \right)' D_X^2 X + \left(\frac{k_2}{k_1} - \frac{k_1'}{k_1^3} \left(\frac{k_1}{k_2} \right)' \right) D_X X - \left(\frac{k_1}{k_2} \right)' X$$

If we use (1) and (3) we get,

$$\left(\frac{k_1}{k_2} \right)' = 0$$

from this, we obtain

$$\left(\frac{k_1}{k_2} \right) = \text{constant}$$

Thus the spacelike curve is a general helix.

Corollary 4.1. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a circular helix if and only if*

$$D_X^3 X - (k_1^2 + k_2^2) D_X X = 0 \tag{6}$$

Proof The proof can easily be seen if we take k_1 and k_2 constant in Theorem 4.1.

Theorem 4.2. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a general helix if and only if*

$$D_X^2 Y - \frac{k_1'}{k_1} D_X Y - (k_1^2 + k_2^2)Y = 0 \tag{7}$$

or

$$D_X^2 Y - \frac{k_2'}{k_2} D_X Y - (k_1^2 + k_2^2)Y = 0 \tag{8}$$

Proof Suppose that $\alpha(t)$ is a general helix. Then from (1) it can easily be seen that given equatios are satisfied. Conversely (7) (or (8)) holds. We show that the spacelike curve is a general helix. Then from (1), we have

$$X = \frac{1}{k_1} D_X Y - \frac{k_2}{k_1} Z \tag{9}$$

differentiating (9) and using (1), we obtain

$$\begin{aligned} D_X X = \; & \frac{1}{k_1}\left\{ D_X^2 Y - \frac{k_1'}{k_1} D_X Y - (k_1^2 + k_2^2)Y \right\} + \\ & k_1 Y - \left(\frac{k_2}{k_1}\right)' Z \end{aligned}$$

If we use (1) and (7) we get,

$$\left(\frac{k_2}{k_1}\right)' = 0$$

from this, we obtain

$$\left(\frac{k_1}{k_2}\right) = \text{constant}$$

Thus the spacelike curve is a general helix.

Corollary 4.2. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a circular helix if and only if*

$$D_X^2 Y - (k_1^2 + k_2^2)Y = 0 \tag{10}$$

Proof The proof can easily be seen if we take k_1 and k_2 constant in Theorem 4.2.

Theorem 4.3. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a general helix if and only if*

$$D_X^3 Z - \frac{3k_1^{'}}{k_1} D_X^2 Z - \left\{ \frac{k_2^{''}}{k_2} - \frac{3(k_1^{'})^2}{k_1^2} + k_1^2 + k_2^2 \right\} D_X Z = 0 \qquad (11)$$

or

$$D_X^3 Z - \frac{3k_2^{'}}{k_2} D_X^2 Z - \left\{ \frac{k_2^{''}}{k_2} - \frac{3(k_2^{'})^2}{k_2^2} + k_1^2 + k_2^2 \right\} D_X Z = 0 \qquad (12)$$

Proof Suppose that $\alpha(t)$ is a general helix. Then from (1) it can easily be seen that given equatios are satisfied. Conversely (11) (or (12)) holds. We show that the spacelike curve is a general helix. Then from (1), we have

$$X = \frac{1}{k_1} D_X Y - \frac{k_2}{k_1} \qquad (13)$$

differentiating (13) and using (1), we obtain

$$D_X X = \frac{1}{k_1 k_2} \left\{ D_X^3 Z - \frac{3k_1^{'}}{k_1} D_X^2 Z - \left\{ \frac{k_2^{''}}{k_2} - \frac{3(k_1^{'})^2}{k_1^2} + k_1^2 + k_2^2 \right\} D_X Z \right\} + \frac{1}{k_1^2} \left(\frac{k_2}{k_1} \right)^{'} D_X^2 Z + \left(\frac{k_1}{k_2} - \frac{k_1^{'}}{k_2^3} \left(\frac{k_2}{k_1} \right)^{'} \right) D_X Z - \left(\frac{k_2}{k_1} \right)^{'} Z$$

If we use (1) and (11) we get,

$$\left(\frac{k_2}{k_1} \right)^{'} = 0$$

from this, we obtain

$$\left(\frac{k_1}{k_2} \right) = \text{constant}$$

Thus the spacelike curve is a general helix.

Corollary 4.3. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a circular helix if and only if*

$$D_X^3 Z - (k_1^2 + k_2^2) D_X Z = 0 \tag{14}$$

Proof The proof can easily be seen if we take k_1 and k_2 constant in Theorem 4.3.

In particular, if the binormal Z is a timelike, we obtain analogous results which are contained in theorem 4.4., 4.5., 4.6. and corollary 4.4., 4.5., 4.6.

Theorem 4.4. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a general helix if and only if*

$$D_X^3 X - \frac{3k_1^{'}}{k_1} D_X^2 X - \left\{ \frac{k_1^{''}}{k_1} - \frac{3(k_1^{'})^2}{k_1^2} - k_1^2 + k_2^2 \right\} D_X X = 0 \tag{15}$$

or

$$D_X^3 X - \frac{3k_2^{'}}{k_2} D_X^2 X - \left\{ \frac{k_1^{''}}{k_1} - \frac{3(k_2^{'})^2}{k_2^2} - k_1^2 + k_2^2 \right\} D_X X = 0 \tag{16}$$

Corollary 4.4. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a circular helix if and only if*

$$D_X^3 X + (k_1^2 - k_2^2) D_X X = 0 \tag{17}$$

Theorem 4.5. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a general helix if and only if*

$$D_X^2 Y - \frac{k_1^{'}}{k_1} D_X Y + (k_1^2 - k_2^2) Y = 0 \tag{18}$$

or

$$D_X^2 Y - \frac{k_2^{'}}{k_2} D_X Y + (k_1^2 - k_2^2) Y = 0 \tag{19}$$

Corollary 4.5. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a circular helix if and only if*

$$D_X^2 Y + (k_1^2 - k_2^2)Y = 0 \tag{20}$$

Theorem 4.6. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a general helix if and only if*

$$D_X^3 Z - \frac{3k_1'}{k_1} D_X^2 Z - \left\{ \frac{k_2''}{k_2} - \frac{3(k_1')^2}{k_1^2} - k_1^2 + k_2^2 \right\} D_X Z = 0 \tag{21}$$

or

$$D_X^3 Z - \frac{3k_2'}{k_2} D_X^2 Z - \left\{ \frac{k_2''}{k_2} - \frac{3(k_2')^2}{k_2^2} - k_1^2 + k_2^2 \right\} D_X Z = 0 \tag{22}$$

Corollary 4.6. *Let $\alpha(t)$ be a spacelike curve with timelike principal normal N in a Lorentzian manifold M_1. $\alpha(t)$ is a circular helix if and only if*

$$D_X^3 Z + (k_1^2 - k_2^2) D_X Z = 0 \tag{23}$$

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