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ON BOOLEAN PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. A Boolean analogue of differential calculus has been developed since the fifties, both for its intrinsic interest and in view of applications; see e.g. [6], [7]. In this paper we solve the partial differential equation $\partial^m f / \partial x_1 \dots \partial x_m = g$ and the partial differential system $\partial f / \partial x_i = g_i$ $(i = 1, \dots, m)$. This generalizes previous results of Bochmann, Ping and Levchenkov.

We need a few prerequisites which refer to an arbitrary Boolean algebra (not necessarily the two-element one).

It is well known that every Boolean algebra can be made into a Boolean ring $(B, +, \cdot, 0, 1)$, which means a commutative unitary ring which is also idempotent ($x^2 = x$) and of characteristic 2 ($x + x = 0$); the ring sum is $x + y = (x \wedge y') \vee (x' \wedge y)$, while $x \cdot y = x \wedge y$.

By a *Boolean function* we mean a polynomial of the Boolean ring B. All the functions dealt with in this paper are Boolean functions of a fixed number n of variables, but some of them may depend actually on fewer variables. The *partial derivatives* of a Boolean function f are defined by

$$\frac{\partial f}{\partial x_i} = f(x_i = 0) + f(x_i = 1) \qquad (i = 1, \dots, n)\,, \tag{1}$$

where we have set

$$f(x_i = a) = f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \qquad (a = 0, 1)\ . \tag{2}$$

The following properties are obvious:

$$\frac{\partial(f+g)}{\partial x_i} = \frac{\partial f}{\partial x_i} + \frac{\partial g}{\partial x_i}\ , \tag{3}$$

$$\frac{\partial gf}{\partial x_i} = g\frac{\partial f}{\partial x_i} \qquad \text{if } g \text{ does not depend on } x_i\ , \tag{4}$$

and the fact that $\partial f/\partial x_i$ *does not depend on* x_i. The identity

$$f(x_1, \ldots, x_n) = x_i\frac{\partial f}{\partial x_i} + f(x_i = 0) \tag{5}$$

is immediately checked for $x_i \in \{0, 1\}$, hence it holds for an arbitrary $x_i \in B$ by the Verification Theorem (see e.g. [6] or [7]). It follows from (1) and (5) that

$$f \text{ does not depend on } x_i \Longleftrightarrow \frac{\partial f}{\partial x_i} = 0\ . \tag{6}$$

The *partial derivatives of higher order* are defined in a natural way. Using the same notation as for the conventional partial derivatives of higher order, we obtain readily

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{7}$$

and in general the order of derivation in

$$\frac{\partial^m f}{\partial x_1 \ldots \partial x_m} \qquad (m \leq n) \tag{8}$$

is immaterial.

Proposition 1. *The Boolean partial differential equation*

$$\frac{\partial^m f}{\partial x_1 \ldots \partial x_m} = g \tag{9}$$

has solutions if and only if g does not depend on the variables $x_1, \ldots, x_m$, in which case the solutions are the functions of the form

$$f = \sum_{S \subset M} h_S \prod_{i \in S} x_i + g x_1 \ldots x_m\ , \tag{10}$$

where $M = \{1, \ldots, m\}$ and the functions h_S $(S \subset M)$ do not depend on the variables $x_1, \ldots, x_m$.

Proof. Taking into account the commutativity of the order of derivation, it follows readily from (9) that g does not depend on the variables $x_1, \ldots, x_m$. Conversely, suppose the latter property holds. Then (10) implies easily (9), via (3) and (4). Finally it remains to prove that every solution of equation (9) is of the form (10). But the following interpolation formula is well known:

$$f = \sum_{S \subseteq M} h_S \prod_{i \in S} x_i \; , \tag{11}$$

where the functions h_S $(S \subseteq M)$ do not depend on the variables $x_1, \ldots, x_m$. Then (11) implies

$$\frac{\partial^m f}{\partial x_1 \ldots \partial x_m} = h_M \; ,$$

therefore $h_M = g$ by (9), and this transforms (11) into (10).

Corollary (Bochmann [1], Ping [4], [5]). *The equation $\partial f / \partial x_i = g$ has solutions if and only if g does not depend on x_i, in which case the solutions are the functions of the form $f = g x_i + h$, where h does not depend on x_i.*

Proposition 2. *The Boolean partial differential system*

$$\frac{\partial f}{\partial x_i} = g_i \qquad (i = 1, \ldots, m) \tag{12}$$

has solutions if and only if the following conditions are fulfilled:

$$g_i \text{ does not depend on } x_i \qquad (i = 1, \ldots, m) \; , \tag{13}$$

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \qquad (i, j = 1, \ldots, m \; ; \; i \neq j) \; , \tag{14}$$

in which case the solutions are the functions of the form

$$f = \sum_{i=1}^{m} g_i x_i + \sum_{k=2}^{m} \sum_{i_1, \ldots, i_k \in M} \frac{\partial^{k-1} g_{i_1}}{\partial x_{i_2} \ldots x_{i_k}} x_{i_1} \ldots x_{i_k} + h \; , \tag{15}$$

where $M = \{1, \ldots, m\}$, the indices $i_1, \ldots, i_k$ in each term of (15) are pairwise distinct and the function h does not depedn on $x_1, \ldots, x_m$.

Comments. 1) Ghilezan [2] proved a similar theorem for pseudo-Boolean partial derivatives, in which the consistency conditions are quite similar to (13) and (14), while the solution is simpler than (15).

2) Levchenkov [3] solved system (12) for $m := n := 2$.

Proof. System (12) implies readily (13) and also (14), via (7). Conversely, suppose (13) and (14) hold. Then it is easily proved by induction that the partial derivative

$$\frac{\partial^{k-1} g_{i_1}}{\partial x_{i_2} \ldots \partial x_{i_k}}$$

does not depend on the chosen permutation $(i_1, \ldots, i_k)$ of the set $\{i_1, \ldots, i_k\}$, therefore the expression (15) is unambiguous. We are going to prove by induction on m that f satisfies system (12) if and only if it is of the form (15).

For $m := 1$ this is true by the above Corollary. The inductive step runs as follows. Split condition (12) into two parts: f satisfies the first $m-1$ equations, and $\partial f/\partial x_m = g_m$. In view of the inductive hypothesis, the former condition is equivalent to f being of the form

$$f = \sum_{i=1}^{m-1} g_i x_i + \sum_{k=2}^{m-1} \sum_{i_1,\ldots,i_k \in M_1} \frac{\partial^{k-1} g_{i_1}}{\partial x_{i_1} \ldots\ x_{i_k}} x_{i_1} \ldots x_{i_k} + h_1 \ , \tag{16}$$

where $M_1 = \{1, \ldots, m-1\}$ and h_1 does not depend on $x_1, \ldots, x_{m-1}$. Now we impose the condition $\partial f/\partial x_m = g_m$ on the function (16):

$$\sum_{i=1}^{m-1} \frac{\partial g_i}{\partial x_m} x_i + \sum_{k=2}^{m-1} \sum_{i_1,\ldots,i_k \in M_1} \frac{\partial^k g_{i_1}}{\partial x_{i_1} \ldots \partial x_{i_k} \partial x_m} x_{i_1} \ldots x_{i_k} + \frac{\partial h_1}{\partial x_m} = g_m \ ,$$

whence the last condition (13) and the above Corollary yield

$$\begin{aligned} h_1 = \ & (g_m + \textstyle\sum_{i=1}^{m-1} (\partial g_i/\partial x_m) x_i \\ + \ & \textstyle\sum_{k=2}^{m-1} \sum_{i_1,\ldots,i_k \in M_1} (\partial^k g_i/\partial x_{i_1} \ldots \partial x_{i_k} \partial x_m) x_{i_1} \ldots x_{i_k}) x_m + \ h \ , \end{aligned} \tag{17}$$

where h does not depend on x_m; as a matter of fact, h does not depend on any of the variables $x_1, \ldots, x_{m_1}, x_m$, because $h = h_1(x_m = 0)$. Summarizing, condition (12) is equivalent to the system (16),(17), which is easily seen to be equivalent with (15).

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