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ON THE TRAPEZOID QUADRATURE FORMULA AND APPLICATIONS

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Abstract. The estimation of the remainder term in trapezoid formula for mappings with bounded variation and for lipschitzian mappings are given. Applications for special means are also pointed out.

1. INTRODUCTION

The following inequality is well known in the literature as the *trapezoid inequality:*

$$\left| \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{1}{12} \| f'' \|_\infty (b-a)^3 \tag{1.1}$$

where the mapping $f : [a,b] \to R$ is supposed to be twice differentiable on the interval (a,b) and having the second derivative bounded on (a,b), that is $\| f'' \|_\infty := \sup_{x\in(a,b)} | f''(x) | < \infty$.

Now, if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a partition of the interval $[a,b]$ and f is as above, then we have the *trapezoid quadrature formula*:

$$\int_a^b f(x)dx = A_T(f,I_h) + R_T(f,I_h) \tag{1.2}$$

where $A_T(f, I_h)$ is the *trapezoid rule*

$$A_T(f, I_h) =: \frac{1}{2}\sum_{i=0}^{n-1}[f(x_i) + f(x_{i+1})]h_i \tag{1.3}$$

and the *remainder term* $R_T(f, I_h)$ satisfies the estimation

$$\mid R_T(f, I_h) \mid \leq \frac{1}{12} \parallel f'' \parallel_\infty \sum_{i=0}^{n-1} h_i^3 \tag{1.4}$$

where $h_i := x_{i+1} - x_i$ for $i = 0, ..., n-1$.

When we have an equidistant partitioning of $[a, b]$ given by

$$I_n : x_i := a + \frac{b-a}{n}i,\ \ i = 0, ..., n \tag{1.5}$$

then we have the formula

$$\int_a^b f(x)dx = A_{T,n}(f) + R_{T,n}(f) \tag{1.6}$$

where

$$A_{T,n}(f) := \frac{b-a}{2n}\sum_{i=0}^{n-1}[f(a + \frac{b-a}{n}i) + f(a + \frac{b-a}{n}(i+1))] \tag{1.7}$$

and the remainder satisfies the estimation

$$\mid R_{T,n}(f) \mid \leq \frac{1}{12} \cdot \frac{(b-a)^3}{n^2} \parallel f'' \parallel_\infty . \tag{1.8}$$

For other trapezoid type's inequalities see the recent book [1].

2. TRAPEZOID INEQUALITY FOR MAPPINGS WITH BOUNDED VARIATION

The following trapezoid inequality for mappings with bounded variation holds:

Theorem 2.1. *Let $f : [a,b] \to R$ be a mapping with bounded variation on $[a,b]$. Then we have the inequality*

$$| \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) | \leq \frac{1}{2}(b-a)V_a^b(f) \tag{2.1}$$

where $V_a^b(f)$ is the total variation of f on the interval $[a,b]$.

The constant $\frac{1}{2}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^b (x - \frac{a+b}{2})df(x) = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx. \tag{2.2}$$

If $p : [a,b] \to R$ is continuous on $[a,b]$ and $v : [a,b] \to R$ is with bounded variation on $[a,b]$, then

$$| \int_a^b p(x)dv(x) | \leq \max_{x \in [a,b]} | p(x) | V_a^b(v). \tag{2.3}$$

Applying the inequality (2.3) for $p(x) = x - \frac{a+b}{2}, v(x) = f(x), x \in [a,b]$, we get

$$| \int_a^b (x - \frac{a+b}{2})df(x) | \leq \max_{x \in [a,b]} | x - \frac{a+b}{2} | V_a^b(f) = \frac{b-a}{2}V_a^b(f). \tag{2.4}$$

and the inequality (2.1) is proved.

Now, assume that the inequality (2.1) holds with a constant $C > 0$,i.e.,

$$| \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) | \leq C(b-a)V_a^b(f). \tag{2.5}$$

Consider the mapping $f:[a,b]\to R$,

$$f(x)=\begin{cases}1 & \text{if} \quad x\in\{a,b\},\\ 0 & \text{if} \quad x\in(a,b).\end{cases}$$

Then f is with bounded variation and we have

$$\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}(b-a)=-(b-a)$$

and

$$V_a^b(f)(b-a)=2(b-a)$$

and then by (2.5) we get

$$b-a\leq 2C(b-a)$$

which implies that $C\geq\frac{1}{2}$ and the sharpness of (2.1) is proved.

The following corollary holds:

Corollary 2.2. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) whose derivative is integrable on (a,b). Then we have the inequality:*

$$\left|\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}(b-a)\right|\leq\frac{1}{2}\parallel f'\parallel_1(b-a). \tag{2.6}$$

Remark 2.3. It is well known that if $f:[a,b]\to R$ is a convex mapping on $[a,b]$, then *Hermite-Hadamard's* inequality holds

$$f(\frac{a+b}{2})\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f:I\subset R\to R$ is convex on I and $a,b\in Int(I), a<b$; then f'_+ is monotonous nondecreasing on $[a,b]$ and by Theorem 2.1 we get

$$0\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{1}{2}\parallel f'_+\parallel_1(b-a) \tag{2.8}$$

which provides a counterpart for the second membership of Hermite-Hadamard's inequality.

The following corollary for trapezoid composite formula holds:

Corollary 2.4. *Let $f : [a,b] \to R$ be a mapping with bounded variation on $[a,b]$ and I_h a partition of $[a,b]$. Then we have the trapezoid quadrature formula (1.2) and the remainder term $R_T(f,I_h)$ satisfies the estimation:*

$$\mid R_T(f,I_h) \mid \leq \frac{1}{2}\gamma(h)V_a^b(f). \tag{2.9}$$

where $\gamma(h) := \max\{h_i | i = 0,...,n-1\}$.
Moreover, the constant $\frac{1}{2}$ is the best possible one.

The case of equidistant partitioning is embodied in the following corollary:

Corollary 2.5. *Let I_n be an equidistant partitioning of $[a,b]$ and f be as in Theorem 2.1. Then we have the formula (1.6) and the remainder satisfies the estimation:*

$$\mid R_{T,n}(f) \mid \leq \frac{1}{2n}(b-a)V_a^b(f). \tag{2.10}$$

Remark 2.6. If we want to approximate the integral $\int_a^b f(x)dx$ by trapeziod formula $A_{T,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := [\frac{1}{2\varepsilon} \cdot (b-a)V_a^b(f)] + 1$$

and $[r]$ denotes the integer part of $r \in R$.

Comments 2.7. If the mapping $f : [a, b] \to R$ is neither twice differentiable nor the second derivative is bounded on (a, b), then we can not apply the classical estimation in trapezoid formula using the second derivative. But if we assume that f is with bounded variation, then we can use instead the formula (2.9).

We give here a class of mappings which are with bounded variation but having the second derivative unbounded on the given interval.

Let $f_{p,q} : [a, b] \to R$, $f_{p,q}(x) := (x^q - a^q)^p$ where $p \in (1, 2)$ and $q \geq 2$. Then obviously

$$f'_{p,q}(x) := pqx^{q-1}(x^q - a^q)^{p-1}, x \in (a, b)$$

and

$$f''_{p,q}(x) = pq\frac{x^{q-2}[(pq-1)x^q - (q-1)a^q]}{(x^q - a^q)^{2-p}}, \; x \in (a, b).$$

It is clear that f is with bounded variation and

$$V_a^b(f) = (b^q - a^q)^p < \infty$$

but $\lim_{x \to a+} f''_{p,q}(x) = +\infty$.

3. TRAPEZOID INEQUALITY FOR LIPSCHITZIAN MAPPINGS

The following trapezoid inequality for lipschitzian mappings holds:

Theorem 3.1. *Let $f : [a, b] \to R$ be an $L-$lipschitzian mapping on $[a, b]$. Then we have the inequality*

$$\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b - a) \right| \leq \frac{1}{4}L(b - a)^2. \tag{3.1}$$

The constant $\frac{1}{4}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^b (x - \frac{a+b}{2}) df(x) = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx. \tag{3.2}$$

If $p : [a,b] \to R$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to R$ is L-lipschitzian on $[a,b]$, then

$$\mid \int_a^b p(x)dv(x) \mid \leq L \int_a^b \mid p(x) \mid dx. \tag{3.3}$$

Applying the inequality (3.3) for $p(x) = x - \frac{a+b}{2}, v(x) = f(x), x \in [a,b]$, we get

$$\mid \int_a^b (x - \frac{a+b}{2}) df(x) \mid \leq L \int_a^b \mid x - \frac{a+b}{2} \mid dx. \tag{3.4}$$

But

$$\int_a^b \mid x - \frac{a+b}{2} \mid dx = \frac{(b-a)^2}{4}$$

and then by (3.4), via the identiry (3.2), we deduce the desired inequality (3.1).

Now, assume that the inequality (3.1) holds with a constant $C > 0$,i.e.,

$$\mid \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) \mid \leq CL(b-a)^2. \tag{3.5}$$

Consider the mapping $f : [a,b] \to R, f(x) = \mid x - \frac{a+b}{2} \mid$. Then

$$\mid f(x) - f(y) \mid = \mid\mid x - \frac{a+b}{2} \mid - \mid y - \frac{a+b}{2} \mid\mid \leq \mid x - y \mid$$

for all $x, y \in [a,b]$; which shows that f is $L-$lipschitzian with the constant $L = 1$. For this mapping we have

$$\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) = -\frac{(b-a)^2}{4}$$

and

$$L(b-a)^2 = (b-a)^2$$

and then by (3.5) we get

$$\frac{(b-a)^2}{4} \leq C(b-a)^2$$

which implies that $C \geq \frac{1}{4}$ and the sharpness of (3.1) is proved.

The following corollary holds:

Corollary 3.2. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b). Then we have the inequality:*

$$| \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) | \leq \frac{1}{4} \| f' \|_\infty (b-a)^2. \tag{3.6}$$

Remark 3.3. It is well known that if $f : [a,b] \to R$ is a convex mapping on $[a,b]$, then *Hermite-Hadamard's* inequality holds (see (2.7)).

Now, if we assume that $f : I \subset R \to R$ is convex on I and $a, b \in Int(I), a < b$; then f'_+ is monotonous nondecreasing on $[a,b]$ and by Theorem 3.1 we get

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{4} f'_+(b)(b-a) \tag{3.7}$$

which provides another counterpart for the second membership of Hermite-Hadamard's inequality.

The following corollary for trapezoid composite formula holds:

Corollary 3.4. *Let $f : [a,b] \to R$ be an $L-$lipschitzian mapping on $[a,b]$ and I_h a partition of $[a,b]$. Then we have the trapezoid quadrature formula (1.2) and the remainder term $R_T(f, I_h)$ satisfies the estimation:*

$$| R_T(f, I_h) | \leq \frac{1}{4} L \sum_{i=0}^{n-1} h_i^2. \tag{3.8}$$

Moreover, the constant $\frac{1}{4}$ *is the best possible one.*

The case of equidistant partitioning is embodied in the following corollary:

Corollary 3.5. *Let* I_n *be an equidistant partitioning of* $[a, b]$ *and* f *be as in Theorem 3.1. Then we have the formula (1.6) and the remainder satisfies the estimation:*

$$| R_{T,n}(f) | \leq \frac{1}{4} \cdot \frac{L}{n}(b-a)^2. \tag{3.9}$$

Remark 3.6. If we want to approximate the integral $\int_a^b f(x)dx$ by trapeziod formula $A_{T,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := [\frac{1}{4} \cdot \frac{L}{\varepsilon}(b-a)^2] + 1.$$

4. APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means:

1. *Arithmetic mean*

$$A = A(a, b) := \frac{a+b}{2}, a, b \geq 0;$$

2. *Geometric mean*

$$G = G(a, b) := \sqrt{ab}, a, b \geq 0;$$

3. *Harmonic mean*

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, a, b > 0;$$

4. *Logarithmic mean*

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a},\ \ a,b>0, a\neq b;$$

5. *Identric mean*

$$I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, a,b>0, a\neq b;$$

6. *p-Logarithmic mean*

$$L_p = L_p(a,b) := \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, p\in R\backslash\{-1,0\}, a,b>0, a\neq b.$$

It is well known that L_p is monotonous nondecreasing over $p \in R$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A. \tag{4.1}$$

In what follows, by the use of Theorem 2.1, we point out some new inequalities for the above means.

1. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = x^p, p \in R\backslash\{-1,0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L_p(a,b), \frac{f(a)+f(b)}{2} = A(a^p,b^p),$$

$$\|f'\|_1 = \mid p \mid (b-a)L_{p-1}^{p-1}, p \in R\backslash\{-1,0,1\}.$$

Using the inequality (2.6) we get

$$\mid L_p^p(a,b) - A(a^p,b^p) \mid \leq \frac{\mid p \mid}{2} L_{p-1}^{p-1}(b-a)^2. \tag{4.2}$$

2. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = \frac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L^{-1}(a,b), \frac{f(a)+f(b)}{2} = H^{-1}(a,b), \|f'\|_1 = \frac{b-a}{G^2(a,b)}.$$

Using the inequality (2.6) we get

$$0 \leq L - H \leq \frac{(b-a)^2}{2G^2} LH. \tag{4.3}$$

3. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = \ln I(a,b), \frac{f(a)+f(b)}{2} = \ln G(a,b), \|f'\|_1 = \frac{b-a}{L(a,b)}.$$

Using the inequality (2.6) we get

$$1 \leq \frac{I}{G} \leq \exp[\frac{(b-a)^2}{2L}]. \tag{4.4}$$

Now, using Theorem 3.1 we can also state the following inequalities:

4. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = x^p, p \in R\backslash\{-1,0\}$. Then

$$\|f'\|_\infty = \delta_p(a,b) := \begin{cases} pb^{p-1} & \text{if} \quad p \geq 1, \\ \mid p \mid a^{p-1} & \text{if} \quad p \in (-\infty, 1)\backslash\{-1,0\}. \end{cases}$$

Using the inequality (3.6) we get

$$\mid L_p^p(a,b) - A(a^p, b^p) \mid \leq \frac{1}{4}\delta_p(a,b)(b-a). \tag{4.5}$$

5. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = \frac{1}{x}$. Then

$$\|f'\|_\infty = \frac{1}{a^2}.$$

Using the inequality (3.6) we get

$$0 \leq L - H \leq \frac{b-a}{4a^2} LH. \tag{4.6}$$

6. Let $f : [a, b] \to R$ $(0 < a < b), f(x) = \ln x$. Then

$$\|f'\|_\infty = \frac{1}{a}.$$

Using the inequality (3.6) we get

$$1 \leq \frac{I}{G} \leq \exp(\frac{b-a}{4a}). \tag{4.7}$$

References

[1] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, 1994.