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H(τ) HAS FUNCTIONAL ATTRIBUTES

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Abstract. The theory of functions of several complex variables which is a natural development of the theory of functions of one complex variable, has come to the fore due to effective applications of the method of this theory in a variety of sciences.

On the theory, Boulami [1] created an algebraic structure, this structure we found to be interesting as it has some attributes in the area of functional analysis when an appropriate metric is defined upon it . This paper is based majorly on the theorems of Hartog.

1. PRELIMINARIES

In an n-dimensional complex space C^n the point $z = (z_1, z_2, \ldots, z_n)$ is n-tuppled and each $z_k = x_{2k-1} + ix_{2k},\ k = 1, 2, ..., n$. The complex space C^n may be interpreted as an ordinary Euclidean space of the real variables $x_j,\ j = 1, 2, ..., 2n$ of dimension $2n$ hence $C^n \equiv R^{2n}$. For this reason, the notions of an open and a closed region, an interior, an exterior and a boundary point, a $\delta - neighbourhood$ of a point z^0 etc, can be explained from R^{2n} point of view . For instance the set $C(\delta, z^0)$ contains points $z \in C^n$ that satisfy the condition:

$$| z_k - {z_k}^0 | < \delta_k, \qquad k = 1, 2, ..., n.$$

where the symbol $\delta = (\delta_1, \delta_2, ..., \delta_n)$ stands for an ordered collection of real numbers $\delta_k > 0$, $D(r, z^0) \equiv C(r, z^0)$. $r = (r_1, r_2, ..., r_n)$, $r_k > 0$ is called a disk centred at $z^0 = ({z_1}^0, {z_2}^0, ..., {z_n}^0)$. Let $\omega = f(z)$ be a function defined on $A \subset C^n$ then $\omega_k = f_k(z_k)$ where z_k is obtained by fixing all z_j, $j = 1, 2, ..., k-1, k+1, ..., n$.

Suppose for arbitrarily fixed values: $z_1^0, z_2^0, ..., z_{k-1}^0, z_{k+1}^0, ..., z_n^0$ each function ω_k is analytic in the corresponding domains $A_k \subset C^n$ then

$$\frac{\partial \omega_k}{\partial z_k} = \lim_{\Delta z_k \to 0} \frac{f(z_1, z_2, ...z_{k-1}, z_k + \Delta z_k, z_{k+1}, ..., z_n) - f(z_1, z_2, ..., z_n)}{\Delta z_k}$$

exists and $\omega = f(z)$ is analytic in C^n (Hartog's). Let τ be a cover of C^n and let $\phi : \tau \to C$ be a function of class C^1 on τ then:

$$d\phi = \sum_{j=1}^{n} \frac{\partial \phi}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial \phi}{\partial \overline{z}_j} d\overline{z}_j$$

where

$$\frac{\partial \phi}{\partial z_j} = \frac{1}{2} \left(\frac{\partial \phi}{\partial x_{2j-1}} - i \frac{\partial \phi}{\partial x_{2j}} \right)$$

and

$$\frac{\partial \phi}{\partial \overline{z}_j} = \frac{1}{2} \left(\frac{\partial \phi}{\partial x_{2j-1}} + i \frac{\partial \phi}{\partial x_{2j}} \right)$$

since

$$2 \frac{\partial \phi}{\partial z_j} = \frac{\partial \phi}{\partial x_{2j-1}} \frac{\partial x_{2j-1}}{\partial z_j} + \frac{\partial \phi}{\partial x_{2j}} \frac{\partial x_{2j}}{\partial z_j},$$

$\frac{\partial z_j}{\partial x_{2j-1}} = 1$ and $\frac{\partial z_j}{\partial x_{2j}} = i$. The set of all analytic function in τ is denoted by $H(\tau)$.

Notations: We adopt the following notations for the sake of brevity:

- By bd we mean the boundary of a polydisk $d = d_1 x d_2 x ... d_n x$ in C^n where each d_i, $i = 1, 2, ..., n$ is a disk in the complex plane C.
- For any $\beta \in N^m$ there are m-tuples that is $\beta = (\beta_1, \beta_2, ...\beta_m) \in N^m$
- $\sum_{j=1}^{m} \beta_j = \beta_1 + \beta_2 + ... + \beta_m$
- $z^\beta = \Pi_{j=1}^{m} z_j^{\beta_j}$
- $\partial^\beta = \frac{\partial^{\sum_{j=1}^{m} \beta_j}}{\Pi_{j=1}^{m} \partial z_j^{\beta}}$

- $D_I = \partial/\partial x_I$ so that $D_{2k-1} = \partial/\partial x_{2k-1}$ and $D_{2k} = \partial/\partial x_{2k}$
- The smallest convex set in $T \subset \tau$ denoted by $\triangle$
- $T \equiv \triangle^n$ and $bT \equiv b\triangle^n$
- Ω is a cover of C such that $\Omega^n \equiv \tau$.

Cauchy's theorem for a triangle. *Suppose $\triangle$ is a closed triangle in a plane open set Ω, $p \in \Omega$, f is continuous on Ω and $f \in H(\Omega - p)$ then*

$$\int_{b\triangle} f(z)dz = 0.$$

Morera's theorem. *Suppose f is a continuous complex function in an open set Ω such that:*

$$\int_{b\triangle} f(z)dz = 0.$$

For every closed triangle $\triangle \subset \Omega$ then $f \in H(\Omega)$.

Remark.

- The above theorems are valid for a single complex variable z. For $z = (z_1, z_2, \ldots, z_n)$, we merely repeat their applications as in:

$$\int_{bT} \phi(z)dz \leq \int_{b\triangle} \int_{b\triangle} ... \int_{b\triangle} f_1(z_1) f_2(z_2) ... f_n(z_n) dz_1 dz_2 ... dz_n = 0$$

- The theorems also apply for $f \in H(\Omega)$. For more explanation as well as the proofs of the theorems see Rudin [6] pp. 221-224.

2. RESULTS

Theorem. *The set $H(\tau)$ is a ring and a vector space over C.*

Proof. Let $f, g \in H(\tau)$ and α, β scalars, then $\alpha f, \beta g \in H(\tau)$ now:

$$\begin{aligned} \alpha f &= \alpha f(z_1, z_2, ..., z_n) \\ &= \alpha F(x_1, x_2, ..., x_{2n}) \\ &= \alpha u^f(x_1, x_3, ..., x_{2k-1}) + i\alpha v^f(x_2, x_4, ..., x_{2k}) \\ &= \alpha u^f + i\alpha v^f, \qquad k = 1, 2, ..., 2n. \end{aligned}$$

similarly, $\beta = \beta u^g + i\beta v^g$ hence:

$$\alpha f + \beta g = \alpha u^f + \beta u^g + i(\alpha v^f + \beta v^g) = U + iV.$$

By the $C - R$ equations;

$$D_{2k-1}\alpha u^f = D_{2k}\alpha v^f, \quad D_{2k}\alpha u^f = -D_{2k-1}\alpha v^f$$

and

$$D_{2k-1}\beta u^g = D_{2k}\beta v^g, \quad D_{2k}\beta u^g = -D_{2k-1}\beta v^g, \quad k = 1, 2, ..., 2n,$$

then;

$$D_{2k-1}U = D_{2k-1}\left(\alpha u^f + \beta u^g\right) = D_{2k}\left(\alpha v^f + \beta v^g\right) = D_{2k}V$$

and

$$D_{2k}U = D_{2k}\left(\alpha u^f + \beta u^g\right) = -D_{2k-1}\left(\alpha v^f + \beta v^g\right) = -D_{2k-1}V$$

the $C - R$ equations are satisfied, consequently; $\alpha f + \beta g = \beta g + \alpha f \in H(\tau)$.

It is easy to see that: $(\alpha + \beta)f = \alpha f + \beta f = f(\alpha + \beta) \in H(\tau)$. As before,

$$h(z) = h(x_i) = U^h + iV^h, \quad i = 1, 2, ..., 2n$$

and

$$g = U^g + iV^g$$

then;

$$gh = (U^g + iV^g)\left(U^h + iV^h\right) = \left(U^gU^h - V^gV^h\right) + i\left(V^gU^h + U^gV^h\right) = U + iV,$$

then

$$D_{2k-1}U = D_{2k-1}(U^gU^h - V^gV^h) = D_{2k}(V^gU^h + U^gV^h) = D_{2k}V.$$

Also

$$D_{2k-1}V = D_{2k-1}(V^gU^h + U^gV^h) = -D_{2k}(U^gU^h - V^gV^h) = -D_{2k}U.$$

It is easy to see that:

$$\alpha f(\beta g) = \beta\alpha g f \in H(\tau)$$

and

$$(g + f)h = gh + fh = h(g + f) \in H(\tau).$$

Thus the proof is established.

Theorem. Let d be a polydisk covering of C^n and let ϕ be a continuous function in $\overline{d}$ which on d is an analytic function relative to each of coordinate z_j when other coordinates are fixed in τ. Then $\phi \in C^\infty \cap H(d)$ and

$$\phi(z) = \frac{1}{2\pi i^n}\oint_{bd}\frac{\phi(\xi_1, \xi_2, ..., \xi_n)}{(\xi_1 - z_1)(\xi_2 - z_2)...(\xi_n - z_n)}d\xi_1 d\xi_2 ... d\xi_n, \quad z \in d.$$

Proof. For every $k = 1, 2, ..., n$ the function $z_k \mapsto \phi(z_1, z_2, ..., z_n)$ is continuous in $\overline{d}_k$ and is analytic in d_k, the other coordinates, i. e., $z_1, z_2, ..., z_{k-1}, z_{k+1}, ..., z_n$ being fixed as $\alpha_1, \alpha_2, ..., \alpha_{k-1}, \alpha_{k+1}, ..., \alpha_n$. By the Cauchy integral formula in one variable, we have:

$$\phi_k(z_1, z_2, ..., \alpha_k, ...z_n) = \frac{1}{2\pi i}\oint_{bd_k}\frac{\phi(\alpha_1, \alpha_2, ..., \alpha_{k-1}, z_k, \alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n)}{(z_k - \alpha_k)}\, dz_k.$$

We have

$$\phi_k(\alpha_1, z_2, ..., z_n) = \frac{1}{2\pi i}\oint_{bd_1}\frac{\phi(z_1, \alpha_2, \alpha_3, ..., \alpha_n)}{(z_1 - \alpha_1)}\, dz_1.$$

Now let $\alpha_3, \alpha_4, ..., \alpha_n$ be constants and vary z_2 whilst $z_1 \in bd_1$, remains *asd* it is, then from the function $z_2 \mapsto \phi(z_1, z_2, \alpha_3, ..., \alpha_n)$ which is analytic in d_2 and continuous in $\overline{d}_2$ we have

$$\phi_k(\alpha_1, \alpha_2, z_3, z_4, ..., z_n) = \frac{1}{2\pi i} \oint_{bd_2} \frac{\phi(z_1, z_2, \alpha_3, \alpha_4, ..., \alpha_n)}{(z_2 - \alpha_2)} \, dz_2.$$

If we continue in this fashion and combine our result, we have

$$\phi_k(\alpha_1, \alpha_2, ..., \alpha_n) = \frac{1}{2\pi i^n} \oint_{bd} \frac{\phi(z_1, z_2, ..., z_n)}{(z_1 - \alpha_1)(z_2 - \alpha_2)...(z_n - \alpha_n)} \, dz_1 dz_2 ... dz_n.$$

By interchanging between α_j and z_j, $j = 1, 2, ..., n$ the theorem is proved .Also the integrand in the above equation is of class C^∞ and it is analytic with respect to z hence $\phi \in C^\infty \cap H(d)$

Corollary. *Let τ be a covering of C^m, then for any compact $T \subset \tau$ and any neighbourhood covering d of T and $\beta = (\beta_1, \beta_2, ..., \beta_m)$ there exist constant C_β such that for all $\phi \in H(\tau)$*

$$lub_T \mid \partial^\beta \phi \mid \leq C_\beta \mid\mid \phi \mid\mid_{L^1(d)} .$$

Proof. Suppose that $d = d_1, d_2, ..., d_n$ covers any polydisk containing T. The proof follows from the repeated use of Cauchy's inequality and the fact that T is covered by a finite number of polydisk each of which is contained in d:

$$lub_T \left| \frac{\partial^\beta \phi}{\partial z_1^{\beta_1}} \right| \leq C_{\beta_1} \oint_{d_1} \mid \phi(z_1, z_2, ..., z_m) \mid \, dx_1 dx_2.$$

Also

$$\begin{aligned} lub_T \left| \frac{\partial \beta_1 + \beta_2 \phi}{\partial z_1^{\beta_1} \partial z_2^{\beta_2}} \right| &\leq C_{\beta_2} \oint_{d_2} \left| \frac{\partial \beta_1}{\partial z_1^{\beta_1}} \phi(z_1, z_2, ..., z_m) \right| \, dx_3 dx_4 \\ &\leq C_{\beta_1} C_{\beta_2} \oint_{d_1 X d_2} \mid \phi(z_1, z_2, ..., z_m) \mid \, dx_1 dx_2 dx_3 dx_4. \end{aligned}$$

continuing in this fashion, we have

$$lub_T \mid \partial^\beta \phi \mid \leq C_\beta \oint_d \mid \phi(z_1, z_2, ..., z_m) \mid \, dx_1 dx_2 ... dx_{2m}.$$

Theorem. *Let (ϕ_m) be a sequence of analytic functions in τ which converges uniformly on a compact subset T of τ to a function ϕ then $\phi \in H(\tau)$.*

Proof. Let $\phi_{m_i} \in H(\tau)$ for $m_i = 1, 2, ...\quad i = 1, 2, ..., n$ and $\phi_{m_i} \to \phi$ uniformly on the compact subset T of τ. Then the convergence is uniform on each compact disc in τ, ϕ is continuous, since T is compact, then

$$\begin{aligned} &\oint_{bT} \phi(z)dz = \lim_{m_i\to\infty} \oint_{bT} \phi_{m_i}(z)dz \\ \leq\ & \lim_{m_1\to\infty} \lim_{m_2\to\infty} \cdots \lim_{m_n\to\infty} \oint_{b\triangle} \oint_{b\triangle} \cdots \oint_{b\triangle} f_1(z_1) f_2(z_2)...f_n(z_n)\, dz_1 dz_2 ... dz_n = 0 \end{aligned}$$

since each

$$lim_{m_i\to\infty} \oint_{b\triangle} f_i(z_i)\, dz_i = 0,$$

by Cauchy's theorem. Hence Morera's theorem implies that $\phi \in H(\tau)$.

Corollary. *By defining the metric:*

$$d(\phi_j, \phi_k) = \begin{cases} 0, \ j = k, \\ \mid \phi_j - \phi_k \mid, \ j \neq k \end{cases}$$

as that induced by the norm

$$\mid\mid \phi \mid\mid_{L^1(d)} \geq \frac{1}{C_\beta} lub_T \mid \partial^\beta \phi \mid .$$

$(H(\tau), d)$ is Banach whilst $(H(T), d)$ is a normed space of complex functions with respect to pointwise addition and scalar multiplication.

Proof. It suffices it to show that

$$\mid\mid \phi \mid\mid_{L^1(d)} \geq \frac{1}{C_\beta} lub_T \mid \partial^\beta \phi \mid$$

induces $d(\phi_j, \phi_k)$, see Simmons [7] p. 81.

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