A NOTE ON GRAPHS WITH TWO MAIN EIGENVALUES

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Abstract. Let $G$ be a simple connected or disconnected graph which has exactly two main eigenvalues. Let $G_k = G \setminus k$ be the corresponding vertex deleted subgraph of $G$. If $G_i$ and $G_j$ are cospectral in this paper we prove that their complementary graphs $\overline{G_i}$ and $\overline{G_j}$ are also cospectral.

Let $G$ be a simple graph of order $n$ with vertex set $V(G) = \{1, 2, \ldots, n\}$. The spectrum of $G$ consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its ordinary adjacency matrix $A$ and is denoted by $\sigma(G)$.

We say that an eigenvalue $\mu$ of $G$ is main if and only if $\langle \mathbf{j}, P \mathbf{j} \rangle = n \cos^2 \alpha \neq 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $P$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $E_A(\mu)$. The quantity $\beta = |\cos \alpha|$ is called the main angle of $\mu$.

Let $A^k = [a_{ij}^{(k)}]$ for any non-negative integer $k$. The number $N_k$ of all walks of length $k$ in $G$ equals $\textbf{sum} \ A^k$, where $\textbf{sum} \ M$ is the sum of all elements in a matrix $M$. 
According to [2], [3], the generating function $H_G(t)$ is defined by $H_G(t) = \sum_{k=0}^{+\infty} N_k t^k$.

Besides, it was proved in [2] that

$$H_G(t) = \frac{1}{t} \left[ \frac{(-1)^n P_G \left( -\frac{t+1}{t} \right)}{P_G \left( \frac{1}{t} \right)} - 1 \right],$$

where $P_G(\lambda) = |\lambda I - A|$ is the characteristic polynomial of $G$ and $\bar{G}$ its complementary graph. We also note that $H_G(t)$ can be represented in the form

$$H_G \left( \frac{1}{\lambda} \right) = \frac{n_1 \lambda}{\lambda - \mu_1} + \frac{n_2 \lambda}{\lambda - \mu_2} + \cdots + \frac{n_k \lambda}{\lambda - \mu_k},$$

where $n_i = n_\beta_i^2$ and $n_1 + n_2 + \cdots + n_k = n$; $\mu_i$ and $\beta_i$ ($i = 1, 2, \ldots, k$) stand for the main eigenvalues and main angles of $G$, respectively. Using this notation we can see that $N_m = n_1 \mu_1^m + n_2 \mu_2^m + \cdots + n_k \mu_k^m$ for any non-negative integer $m$.

In [1] was proved that the graph $G$ and its complement $\bar{G}$ have the same number of main eigenvalues. We also know that $\lambda_1(G) + \lambda_1(\bar{G}) = n - 1$ if and only if $G$ is regular. More generally, it was proved in [4] the following result.

**Theorem 1.** Let $\mu_1, \mu_2, \ldots, \mu_k$ and $\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_k$ be the main eigenvalues of the graph $G$ and its complement $\bar{G}$, respectively. Then $\sum_{i=1}^{k} (\mu_i + \bar{\mu}_i) = n - k$.

Let $G = G_1 \cup G_2$ be the union of two regular graphs $G_1$ and $G_2$ (not necessarily connected) of order $n_1$ and $n_2$ and degree $r_1$ and $r_2$ ($r_1 \neq r_2$), respectively. Using the fact that $H_G(t) = H_{G_1}(t) + H_{G_2}(t)$, we have $H_G \left( \frac{1}{\lambda} \right) = \frac{n_1 \lambda}{\lambda - r_1} + \frac{n_2 \lambda}{\lambda - r_2}$, wherefrom we obtain that $G$ has two main eigenvalues $r_1$ and $r_2$.

Let $\bar{\tau}_1, \bar{\tau}_2$ denote the main eigenvalues of $\bar{G}$ and let $\bar{n}_1 = n_1 \bar{\beta}_1^2$, $\bar{n}_2 = n_2 \bar{\beta}_2^2$, where $\bar{\beta}_1$ and $\bar{\beta}_2$ are the main angles of $\tau_1$ and $\tau_2$, respectively. For the graph $\bar{G}$ let $\bar{A}^k = [\bar{a}_{ij}^{(k)}]$ for any non-negative integer $k$, where $\bar{A}$ is the adjacency matrix of $\bar{G}$.

In the sequel, we shall use the following notations: $\bar{a}_{ij}^{(k,1)} = \bar{a}_{ij}^{(k)}$ if $i, j \in V(G_1)$; $\bar{a}_{ij}^{(k,2)} = \bar{a}_{ij}^{(k)}$ if $i, j \in V(G_2)$; and $\omega_{ij}^{(k)} = \bar{a}_{ij}^{(k)}$, otherwise. We can see that $\omega_{ij}^{(k)}$ is independent of the choice of vertices $i$ and $j$. Consequently, in order to simplify the
notation, we shall set $\omega^{(k)}_{ij} = \omega_k$. Let

$$\Delta_{k,\ell} = \frac{s_{\ell}^k + (-1)^{k-1}(r_{\ell} + 1)^k}{n_{\ell}} + n_\tau \left[ \sum_{m=0}^{k-2} s_{\ell}^m \omega_{k-1-m} \right],$$

where $s_{\ell} = (n_{\ell} - 1) - r_{\ell}$ and $\ell = 3 - \ell$ for $\ell = 1, 2$. By induction on $k$ we can easily see that

$$\bar{a}_{ij}^{(k,\ell)} = \Delta_{k,\ell} + (-1)^k \sum_{m=0}^{k} \binom{k}{m} a_{ij}^{(m)} \quad (i, j \in V(G_{\ell})),

\text{by understanding that } a_{ij}^{(0)} = \delta_{ij}, \text{ where } \delta_{ij} \text{ is the Kronecker delta symbol. Since the proof of relation (4) is trivial it will be omitted. We now note that}

$$\sum_{\ell=1}^{2} \left[ \sum_{i \in V_{\ell}} \sum_{j \in V_{\ell}} \bar{a}_{ij}^{(k,\ell)} \right] + 2n_1n_2\omega_k = n_1\bar{r}_1^k + n_2\bar{r}_2^k,

\text{for any non-negative integer } k. \text{ By a straightforward calculation, it is not difficult to show that the expression for}

\begin{align*}
\bar{r}_1 &= \frac{(n_1n - 2n_1 + n) + (n - n_1 - n_1) r_1 + (n_1 - n_1) r_2}{2n_1 - n} \quad (6)
\end{align*}

\text{and}

\begin{align*}
\bar{r}_2 &= \frac{(n_1n - 2n_1 + n - n^2) + (n_1 - n_1) r_1 + (n_1 - n_1) r_2}{2n_1 - n} \quad (7)
\end{align*}

\text{can be obtained by solving the following system of equations}

$$n_1s_1 + n_2s_2 + 2n_1n_2 = n_1\bar{r}_1 + n_2\bar{r}_2,$$

$$r_1 + r_2 + \bar{r}_1 + \bar{r}_2 = n - 2.$$
Hence,\
\[ n_{1,2} = \frac{n}{2} \pm \frac{n^2 + (n - 2n_1)(r_1 - r_2)}{2\sqrt{\Delta}}, \] (8)
where \( \Delta = (r_1 - r_2 + n)^2 - 4n_1(r_1 - r_2) \). Substituting \( n_1 \) back into (6) and (7), we obtain that\
\[ \bar{r}_{1,2} = \frac{n - 2 - r_1 - r_2}{2} \pm \frac{\sqrt{\Delta}}{2}. \] (9)

Next, we have\
\[ n_1 \omega_k = \sum_{i \in V_1} \left[ \sum_{j \in V_1} a_{ij}^{(k-1,1)} \right] + n_1 s_2 \omega_{k-1}; \]
\[ n_2 \omega_k = \sum_{i \in V_2} \left[ \sum_{j \in V_2} a_{ij}^{(k-1,2)} \right] + n_2 s_1 \omega_{k-1}, \]
from which an easy calculation yields \( n \omega_k + \left[ n_1 r_2 + n_2 r_1 + n \right] \omega_{k-1} = \bar{N}_{k-1} \), where \( \bar{N}_k = \text{sum} \bar{A}^k \). From the last difference equation, we get\
\[ \omega_k = \frac{1}{n} \sum_{i=0}^{k-1} (-1)^{k-1-i} \left( \frac{n}{\nabla_1} \right)^{k-1-i} \bar{N}_i, \]
where \( \nabla = [n_1 r_2 + (n - n_1) r_1 + n] \). Since \( \bar{N}_k = \bar{n}_1 \bar{r}_1 + \bar{n}_2 \bar{r}_2 \) the previous relation is transformed into\
\[ \omega_k = \frac{(-\nabla)^{k-1}}{n^k} \sum_{i=0}^{k-1} (-1)^i \left[ \bar{n}_1 \left( \frac{n \bar{r}_1}{\nabla} \right)^i + \bar{n}_2 \left( \frac{n \bar{r}_2}{\nabla} \right)^i \right] \]
\[ = \frac{(-1)^{k-1}}{n^k} \sum_{\ell=1}^2 \bar{n}_\ell \left[ \nabla^{(k)} + \frac{(-1)^{k+1} n^k \bar{n}_\ell}{\nabla + n \bar{n}_\ell} \right] \]
\[ = \frac{\nabla^{(1)} + \nabla^{(2)}}{n^k \left[ \nabla + n \bar{n}_1 \right] \left[ \nabla + n \bar{n}_2 \right]}, \]
where \( \nabla^{(1)} = (-1)^{k-1} n \nabla^k \left[ \nabla + \bar{n}_1 \bar{r}_2 + \bar{n}_2 \bar{r}_1 \right] \) and \( \nabla^{(2)} = n^k \left[ \nabla \bar{N}_k + n \bar{r}_1 \bar{r}_2 \bar{N}_{k-1} \right] \). We can easily verify that \( \nabla^{(1)} = 0 \) and \( \nabla^{(2)} \) is equal to the right-hand side of (2) and (3), which results in\
\[ \omega_k = \left[ \frac{\bar{n}_1 \bar{r}_1}{\nabla + n \bar{n}_1} \right] \bar{r}^{k-1}_1 + \left[ \frac{\bar{n}_2 \bar{r}_2}{\nabla + n \bar{n}_2} \right] \bar{r}^{k-1}_2. \]
By a straightforward calculation we find that
\[
\left[ \frac{\overline{n}_\ell \overline{n}_\ell}{\sqrt{n} + n \overline{n}_\ell} \right] = \frac{1}{2} \left[ 1 \pm \frac{n - 2 - r_1 - r_2}{\sqrt\Delta} \right],
\]
where ‘+’ and ‘−’ are related to \( \ell = 1 \) and \( \ell = 2 \), respectively. Using the last relation, we finally have
\[
\omega_k = \frac{\overline{r}_1^k - \overline{r}_2^k}{\sqrt{\Delta}} \quad (k = 0, 1, 2, \ldots).
\]

With regard to (3) and (10), we notice that \( \Delta_{k,\ell} \) may be written in the following form:
\[
\Delta_{k,\ell} = \frac{s_1^k + (-1)^{k-1}(r_1 + 1)^k}{n_\ell} + \frac{n_\ell}{\sqrt{\Delta}} \left[ \frac{\overline{r}_1^k - s_1^k}{\overline{r}_1 - s_1} - \frac{\overline{r}_2^k - s_2^k}{\overline{r}_2 - s_2} \right].
\]

Further, let \( S \) be any (possibly empty) subset of the vertex set \( V(G) \) and let \( G_S \) be the graph obtained from the graph \( G \) by adding a new vertex \( x (x \notin V(G)) \), which is adjacent exactly to the vertices from \( S \).

For a square matrix \( M \) denote by \( \{M\} \) the adjoint of \( M \) and for any two subsets \( X, Y \subseteq V(G) \) define \( \langle X, Y \rangle = \sum_{i \in X} \sum_{j \in Y} A_{ij} \), where \( A = [A_{ij}] = \{\lambda I - A\} \). The expression \( \langle X, Y \rangle \) is called the formal product of the sets \( X \) and \( Y \), associated with the graph \( G \). For any two disjoint subsets \( X, Y \subseteq V(G) \) let \( X + Y \) denote the union of \( X \) and \( Y \). Then \( \langle X + Y, Z \rangle = \langle X, Z \rangle + \langle Y, Z \rangle \) for any \( Z \subseteq V(G) \) and \( \langle X, Y \rangle = \langle Y, X \rangle \) for any (not necessarily disjoint) \( X, Y \subseteq V(G) \). According to [5],
\[
P_G(S)(\lambda) = P_G(\lambda) \left[ \lambda - \frac{1}{\lambda} \right] \mathfrak{f}_S\left(\frac{1}{\lambda}\right) \quad \text{and} \quad \langle S, S \rangle = \frac{P_G(\lambda)}{\lambda} \mathfrak{f}_S\left(\frac{1}{\lambda}\right),
\]
where \( \mathfrak{f}_S(t) = \sum_{k=0}^{+\infty} d^{(k)} t^k \) and \( d^{(k)} = \sum_{i \in S} \sum_{j \in S} a_{ij}^{(k)} \) \((k = 0, 1, 2, \ldots)\). More generally, we proved in [6] that
\[
\langle X, Y \rangle = \frac{P_G(\lambda)}{\lambda} \mathfrak{f}_{X,Y}\left(\frac{1}{\lambda}\right) \quad \langle X, Y \subseteq V(G) \rangle,
\]
where \( \mathfrak{f}_{X,Y}(t) = \sum_{k=0}^{+\infty} e^{(k)} t^k \) and \( e^{(k)} = \sum_{i \in X} \sum_{j \in Y} a_{ij}^{(k)} \) \((k = 0, 1, 2, \ldots)\). Setting \( S^\bullet = V(G) \) we obtain that \( \langle S^\bullet, S^\bullet \rangle = \text{sum} \{\lambda I - A\} \) and \( \mathfrak{f}_{S^\bullet}(t) = H_G(t) \). We also note from (12) that for any \( S \subseteq V(G) \),
\[
P_G(S)(\lambda) = \lambda P_G(\lambda) - \langle S, S \rangle,
\]
where \( \langle S, S \rangle \) is the formal product associated with \( G \).
Let $i$ be a fixed vertex from the vertex set $V(G)$ and let $G^i = G \setminus i$ be its corresponding vertex deleted subgraph.

**Proposition 1 (Lepović [9]).** Let $G$ be a connected or disconnected regular graph of order $n$ and degree $r$. Then for any $i \in V(G)$ and any $S \subseteq V(G)$ we have:

\begin{align}
(1^0) \quad P_{G^i} (\lambda) &= \frac{(-1)^{n-1}}{\lambda + r + 1} \left[ (\lambda - \overline{r}) P_G (\overline{\lambda}) - \frac{P_G (\overline{\lambda})}{\lambda + r + 1} \right]; \\
(2^0) \quad P_{G^i} (\lambda) &= P_G (\lambda) - \frac{n - 2 |S|}{\lambda - r} P_G (\lambda); \\
(3^0) \quad P_{G^i} (\lambda) &= \frac{(-1)^{n+1}}{\lambda + r + 1} \left[ (\lambda - \overline{r}) P_G (\overline{\lambda}) + \frac{\lambda + r + 1 - |S|}{\lambda + r + 1} P_G (\overline{\lambda}) \right],
\end{align}

where $\overline{r} = (n - 1) - r$, $\overline{\lambda} = -\lambda - 1$ and $T = V(G) \setminus S$.

Let $G$ be the union of any $k$ (not necessarily connected) graphs $G^{(1)}, G^{(2)}, \ldots, G^{(k)}$ and let $S = S_1 \cup S_2 \cup \cdots \cup S_k \subseteq V(G)$, where $S_i \subseteq V(G^{(i)})$ for $i = 1, 2, \ldots, k$. In [7] it was proved that

\begin{equation}
P_{G^i} (\lambda) = \sum_{i=1}^{k} \left[ P_{G^{(i)}} (\lambda) \prod_{j \in V_i} P_{G^{(j)}} (\lambda) \right] - (k - 1) \lambda \prod_{i=1}^{k} P_{G^{(i)}} (\lambda),
\end{equation}

where $V_i = \{1, 2, \ldots, k\} \setminus \{i\}$. Using the last relation and Proposition (2$^0$), we easily obtain the following result.

**Proposition 2.** Let $G$ be the union of $k$ regular graphs $G_1, G_2, \ldots, G_k$ of order $n_1, n_2, \ldots, n_k$ and degree $r_1, r_2, \ldots, r_k$, respectively. Then for any $S = \bigcup_{m=1}^{k} S_m \subseteq V(G)$, we have

\begin{equation}
P_{G^i} (\lambda) = P_G (\lambda) - \left[ \sum_{m=1}^{k} \frac{n_m - 2 |S_m|}{\lambda - r_m} \right] P_G (\lambda),
\end{equation}

where $T = V(G) \setminus S$ and $S_m \subseteq V(G_m)$ for $m = 1, 2, \ldots, k$.

Let $\mathcal{M}(G) = \{\mu_1, \mu_2, \ldots, \mu_k\}$ be the set of all main eigenvalues of a graph $G$ of order $n$. As is known, if $\lambda \in \sigma(G) \setminus \mathcal{M}(G)$ then $-\lambda - 1 \in \sigma(\overline{G}) \setminus \mathcal{M}(\overline{G})$, which provides the following relation

\begin{equation}
\left[ \prod_{m=1}^{k} (\lambda + \mu_m + 1) \right] P_{\overline{G}} (\lambda) = \left[ \prod_{m=1}^{k} (\lambda - \overline{\mu}_m) \right] (-1)^n P_G (-\lambda - 1),
\end{equation}

where $\overline{\mu}_m = 2r_m$ for all $m$. \hfill $\square$
where \( \overline{m} \in M(G) \) for \( m = 1, 2, \ldots, k \).

Further, let \( x_1, x_2, \ldots, x_n \) denote a complete set of mutually orthogonal normalized eigenvectors of the adjacency matrix \( A \) corresponding to the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of \( G \), respectively. Let \( X = [x_1, x_2, \ldots, x_n] = [x_{ij}] \) denote the orthogonal matrix of eigenvectors \( x_1, x_2, \ldots, x_n \). Besides, let \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \) be the diagonal matrix of the eigenvalues of \( G \). Since \( X^{-1} = X^\top \) and \( A^k = X \Lambda^k X^\top \), where \( X^\top \) is the transpose of \( X \), we have

\[
a^{(k)}_{ij} = \sum_{\nu=1}^{n} x_{i\nu} x_{j\nu} \lambda^k_{\nu} \quad (i, j = 1, 2, \ldots, n),
\]

for any non-negative integer \( k \).

**Proposition 3.** Let \( G \) be the union of two regular graphs \( G_1 \) and \( G_2 \) of order \( n_1 \) and \( n_2 \) and degree \( r_1 \) and \( r_2 \), respectively. Then for any \( S = S_1 \cup S_2 \subseteq V(G) \) we have

\[
(-1)^{n+1} P^{\overline{G}_S}(\lambda) = \left[ 1 - \sum_{m=1}^{2} \frac{n_m}{\lambda + r_m + 1} \right] P^{G_1}_S(\overline{\lambda}) + \left[ \sum_{m=1}^{2} \frac{|S_m|}{\lambda + r_m + 1} \right]^2 P^{G_2}(\overline{\lambda})
\]

\[+ \left[ 1 - \sum_{m=1}^{2} \frac{2|S_m|}{\lambda + r_m + 1} \right] P^{G}(\overline{\lambda}),
\]

where \( \overline{\lambda} = -\lambda - 1 \) and \( S_m \subseteq V(G_m) \) for \( m = 1, 2 \).

**Proof.** Using (12) and (18) we have \( \mathcal{F}_S(\lambda) = \lambda \left[ \sum_{\nu=1}^{n} \frac{d_{\nu}}{\lambda - \lambda_{\nu}} \right] \) where \( d_{\nu} = \sum_{i \in S} \sum_{j \in S} a^{(k)}_{ij} \). Then we easily obtain

\[
\sum_{i=1}^{n} \frac{d_i}{\lambda + \lambda_i + 1} = (\lambda + 1) + \frac{P^{G_1}_S(\overline{\lambda})}{P^{G_2}(\overline{\lambda})}.
\]

Next, denote the formal generating function of \( \overline{G}_S \) by \( \mathcal{F}_{\overline{G}_S}(t) = \sum_{k=0}^{+\infty} d^{(k)} t^k \), where \( d^{(k)} = \sum_{i \in S} \sum_{j \in S} a^{(k)}_{ij} \). Since \( \langle S, S \rangle = \langle S_1, S_1 \rangle + 2 \langle S_1, S_2 \rangle + \langle S_2, S_2 \rangle \) and \( \sqrt{\Delta} = r_1 - r_2 \), using equations (4), (10), (11), (13), (18), we get
In view of (12), (14), (17), (19) and the previous relation, a straightforward calculation yields

\[ (-1)^{n+1} P_{G_S}(\lambda) = \left[ \left( \sum_{m=1}^{2} \frac{n_m |S_m|}{|\lambda - s_m|} \right) + 2 |S_1||S_2| \right] \frac{P_G(\overline{\lambda})}{(\lambda + r_1 + 1)(\lambda + r_2 + 1)} \]

\[ + \frac{(\lambda - \overline{\tau}_1)(\lambda - \overline{\tau}_2)}{(\lambda + r_1 + 1)(\lambda + r_2 + 1)} \left[ P_G(\overline{\lambda}) + P_{G_S}(\overline{\lambda}) \right] + \]

\[ + \sum_{m=1}^{2} \frac{|S_m|}{|\lambda - s_m|} \frac{P_G(\overline{\lambda})}{(\lambda + r_m + 1)} \].

Since \((\lambda - s_1)(\lambda - s_2) - (\lambda - \overline{\tau}_1)(\lambda - \overline{\tau}_2) = n_1n_2\) the last relation is transformed in the form

\[ (-1)^{n+1} P_{G_S}(\lambda) = \left[ 1 - \sum_{m=1}^{2} \frac{n_m}{\lambda + r_m + 1} \right] \left( P_{G_S}(\overline{\lambda}) + P_G(\overline{\lambda}) \right) \]

\[ + \left[ \sum_{m=1}^{2} \frac{|S_m|}{\lambda + r_m + 1} \right] ^2 P_G(\overline{\lambda}) \].

Finally, since \(P_{G_S}(\lambda) = P_{G_T}(\lambda)\) where \(T = T_1 \cup T_2\) and \(T_m = V(G_m) \setminus S_m\), applying (16) to the previous relation we obtain that

\[ (-1)^{n+1} P_{G_S}(\lambda) = \left[ 1 - \sum_{m=1}^{2} \frac{n_m}{\lambda + r_m + 1} \right] \left( \left( \sum_{m=1}^{2} \frac{n_m - 2|S_m|}{\lambda + r_m + 1} \right) P_G(\overline{\lambda}) + \right. \]

\[ + P_{G_S}(\overline{\lambda}) + P_G(\overline{\lambda}) \left. \right] + \left[ \sum_{m=1}^{2} \frac{n_m - |S_m|}{\lambda + r_m + 1} \right] ^2 P_G(\overline{\lambda}) , \]

from which we find the proof.

Let \(G\) be a graph with \(k\) main eigenvalues \(\mu_1, \mu_2, \ldots, \mu_k\) and let \((x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)})\) denote the eigenvector of \(\mu_m\) so that \(\sum_{i=1}^{n} x_i^{(m)} = \sqrt{n_m}\).
Proposition 4 (Lepović [10]). Let $G$ be a connected or disconnected graph of order $n$ with exactly two main eigenvalues $\mu_1$ and $\mu_2$. Then $x_i^{(1)} = \frac{\deg(i) - \mu_2}{\sqrt{\mu_1 - \mu_2}}$ for $i = 1, 2, \ldots, n$.

Proposition 5 (Lepović [10]). Let $G$ be any connected or disconnected graph of order $n$ with $k$ main eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$. Then for any $i \in V(G)$ and any $S \subseteq V(G)$ we have:

$$(-1)^{n+1} P_{G \setminus T}(\lambda) = \left[1 + \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\right] (P_{G_{S}}(\lambda) + P_G(\lambda)) + \left[\sum_{m=1}^{k} \frac{|S_m|}{\lambda - \mu_m}\right]^2 P_G(\lambda);$$

$$(-1)^{n+1} P_{G \setminus S}(\lambda) = \left[1 + \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\right] P_{G_{S}}(\lambda) + \left[1 + \sum_{m=1}^{k} \frac{|S_m|}{\lambda - \mu_m}\right] P_G(\lambda);$$

$$(-1)^{n-1} P_{G'}(\lambda) = \left[1 + \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\right] P_{G'_{S}}(\lambda) - \left[\sum_{m=1}^{k} \frac{|S_m|^{(i)}}{\lambda - \mu_m}\right]^2 P_G(\lambda),$$

where $\lambda = -\lambda - 1$ and $T = V(G) \setminus S$; $|S_m| = \sqrt{n_m} \left[\sum_{i \in S} x_i^{(m)}\right]$ and $|S_m|^{(i)} = \sqrt{n_m} x_i^{(m)}$.

Theorem 2. Let $G$ be a connected or disconnected graph with exactly two main eigenvalues and let $P_{G'}(\lambda) = P_{G'}(\lambda)$. Then $P_{G'}(\lambda) = P_{G'}(\lambda)$.

Proof. According to Proposition it suffices to show that $|\Pi_{1}^{(i)}| = |\Pi_{1}^{(j)}|$ and $|\Pi_{2}^{(i)}| = |\Pi_{2}^{(j)}|$. We note that $|\Pi_{1}^{(i)}| + |\Pi_{2}^{(i)}| = |\Pi_{1}^{(j)}| + |\Pi_{2}^{(j)}|$ (see also [10]). Since $\deg(i) = \deg(j)$ from Proposition it follows that $|\Pi_{1}^{(i)}| = |\Pi_{1}^{(j)}|$, which provides the proof.

Further, for any $S \subseteq V(G)$ denote by $G_{S,T}$ the graph obtained from $G$ by adding two new non-adjacent vertices $x, y$, so that $x$ is adjacent exactly to the vertices from $S$, and $y$ is adjacent exactly to the vertices from $T = V(G) \setminus S$. Besides, let $G_{S,T}$ be the overgraph of $G$ obtained by adding two new adjacent vertices $x, y$, so that $x$ and $y$ are adjacent in $G$ exactly to the vertices from $S$ and $T$, respectively.
**Theorem 3** (Lepović [7]). Let $G$ be any graph of order $n$. Then for any $S \subseteq V(G)$ we have:

$$P_{G,S,T}(\lambda) = \lambda P_{G,S}(\lambda) + (-1)^n P_{\overline{G},S}(-\lambda - 1) - (\lambda^2 + \lambda) P_G(\lambda) +$$

$$+ (-1)^n(\lambda + 1) P_{\overline{G}}(-\lambda - 1) + (\lambda + 1) P_G(\lambda);$$

$$P_{G,S,T}(\lambda) = (\lambda - 1) P_{G,S}(\lambda) + (-1)^n P_{\overline{G},S}(-\lambda - 1) - (\lambda^2 - \lambda) P_G(\lambda) +$$

$$+ (-1)^n \lambda P_{\overline{G}}(-\lambda - 1) + \lambda P_G(\lambda),$$

where $T = V(G) \setminus S$.

**Proposition 6.** Let $G$ be a connected or disconnected graph with $k$ main eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$. Then for any $S \subseteq V(G)$, we have:

$$P_{G,S,T}(\lambda) = \left[2\lambda - \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\right] P_{G,S}(\lambda) - \left[\lambda - \sum_{m=1}^{k} \frac{|S_m|}{\lambda - \mu_m}\right]^2 P_G(\lambda);$$

$$P_{G,S,T}(\lambda) = \left[2(\lambda - 1) - \sum_{m=1}^{k} \frac{n_m}{\lambda - \mu_m}\right] P_{G,S}(\lambda) - \left[(\lambda - 1) - \sum_{m=1}^{k} \frac{|S_m|}{\lambda - \mu_m}\right]^2 P_G(\lambda),$$

where $T = V(G) \setminus S$ and $|S_m| = \sqrt{n_m \left[\sum_{i \in S} x_i^{(m)}\right]}$ for $m = 1, 2, \ldots, k$.

**Proof.** Using Proposition 5 and Theorem 3 by an easy calculation we obtain the required statement.

**References**


