THE STAR IS THE TREE WITH GREATEST GREATEST LAPLACIAN EIGENVALUE

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Abstract. It is shown that among all trees with a fixed number of vertices the star has the greatest value of the greatest Laplacian eigenvalue.

INTRODUCTION

Throughout this paper \( n \) denotes an integer greater than 1. All square matrices are assumed to be of order \( n \) and all vectors are assumed to be column-vectors of dimension \( n \). If \( \vec{C} \) is a column-vector, then \( \vec{C}^t \) is its transpose, which is a row-vector of dimension \( n \). The sum of all the \( n \) components of the vector \( \vec{C} \) is denoted by \( \sigma(\vec{C}) \).

By \( I \) we denote the unit matrix and by \( J \) the square matrix whose all elements are unity. By \( \vec{0} \) and \( \vec{j} \) we denote the vector whose all components are respectively equal to zero and unity.

Let \( G \) be a graph on \( n \) vertices. Label the vertices of \( G \) by \( v_1, v_2, \ldots, v_n \). Then the adjacency matrix \( A(G) \) of \( G \) is a square matrix of order \( n \), defined so that its \((i, j)\)-entry is unity if the vertices \( v_i \) and \( v_j \) are adjacent and is zero otherwise.
The number of first neighbors of a vertex $v$ is the degree of this vertex and is denoted by $d(v)$. Note that if $v_i$ is a vertex of the graph $G$, then $d(v_i)$ is equal to the sum of the $i$-th row of the adjacency matrix $A(G)$.

Let $D(G)$ be a square matrix whose diagonal entries are $d(v_1), d(v_2), \ldots, d(v_n)$ and the off-diagonal elements are zero.

Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of $G$.

The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $L(G)$ are called the Laplacian eigenvalues of the graph $G$ and the respective eigenvectors $\vec{C}_1, \vec{C}_2, \ldots, \vec{C}_n$ the Laplacian eigenvectors of the graph $G$. Thus the equality $L(G) \vec{C}_i = \mu_i \vec{C}_i$ is obeyed for all $i = 1, 2, \ldots, n$. We label the Laplacian eigenvalues so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Details of the theory of Laplacian spectra of graphs can be found in some of the numerous reviews published on this topic [2, 3, 4]. Here we only mention that it is always $\mu_n = 0$.

**SOME AUXILIARY RESULTS**

**Lemma 1.** The vector $\vec{j}$ is a Laplacian eigenvector of any $n$-vertex graph, corresponding to the eigenvalue $\mu_n = 0$.

**Proof.** Because the sum of any row of $L(G)$ is equal to zero, $L(G) \vec{j} = \vec{0} = 0 \cdot \vec{j}$.

In view of Lemma 1 we may choose $\vec{C}_n = \vec{j}$. In what follows we assume that the Laplacian eigenvectors $\vec{C}_i$, $i = 1, 2, \ldots, n - 1$, are orthogonal to $\vec{C}_n$. If $\mu_{n-1} \neq 0$ then this orthogonality condition is automatically satisfied. If however, $\mu_{n-k} = 0$ for some $k \geq 1$, then the requirement that the eigenvectors $\vec{C}_{n-1}, \ldots, \vec{C}_{n-k}$ are chosen to be orthogonal to $\vec{C}_n$ must be additionally stipulated. Recall that $\mu_{n-1} \neq 0$ if and only if the graph $G$ is connected [2, 3, 4].

From the fact that for any vector $\vec{C}$ the scalar product $\vec{j} \cdot \vec{C}$ is equal to $\sigma(\vec{C})$, it follows:

**Lemma 2.** For any graph $G$ and any $1 \leq i \leq n - 1$, $\sigma(\vec{C}_i) = 0$. 
Denote by $\bar{G}$ the complement of the graph $G$ and its Laplacian eigenvalues by $\bar{\mu}_1 \geq \bar{\mu}_2 \geq \cdots \geq \bar{\mu}_{n-1} \geq \bar{\mu}_n = 0$.

**Lemma 3.** If $\bar{C}_i$ is a Laplacian eigenvector of the graph $G$, then $\bar{C}_i$ is a Laplacian eigenvector of the graph $\bar{G}$.

**Proof.** For $i = n$ Lemma 3 follows from Lemma 1. Assume, therefore, that $1 \leq i \leq n - 1$. Then, by Lemma 2, $\sigma(\bar{C}_i) = 0$.

From the construction of the complement of a graph it is clear that $L(G) + L(\bar{G}) = nI - J$. Consequently,

$$L(\bar{G}) \bar{C}_i = [nI - J - L(G)] \bar{C}_i = nI \bar{C}_i - J \bar{C}_i - L(G) \bar{C}_i = n \bar{C}_i - \sigma(\bar{C}_i) \bar{j} - \mu_i \bar{C}_i = (n - \mu_i) \bar{C}_i$$

This not only proves that $\bar{C}_i$ is an eigenvalue of $L(\bar{G})$, but also shows the way in which the Laplacian eigenvalues of $G$ and $\bar{G}$ are related:

**Lemma 4.** For $i = n$, $\bar{\mu}_i = \mu_i = 0$. For $i = 1, 2, \ldots, n - 1$, $\bar{\mu}_i = n - \mu_{n-i}$.

As a direct consequence of Lemma 4 we have

**Lemma 5.** If $G$ is not connected, then $\mu_1 = n$. If $G$ is connected, then $\mu_1 < n$.

The Lemmas 4 and 5 are previously known results [2, 3, 4].

THE MAIN RESULT AND ITS PROOF

A tree is a connected acyclic graph. The star $S_n$ is the $n$-vertex tree in which $n - 1$ vertices are of degree 1 and one vertex is of degree $n - 1$.

**Lemma 6.** For any $n \geq 2$, the greatest Laplacian eigenvalue of the $n$-vertex star is equal to $n$. 
Proof. The complement of the star $S_n$ is disconnected (and consists of the complete graph on $n - 1$ vertices and an isolated vertex). Thus Lemma 6 follows from Lemma 5.

**Lemma 7.** If $T$ is any $n$-vertex tree, not isomorphic to $S_n$, then $\bar{T}$ is connected.

**Proof.** A graph is connected if there is a path between any two of its vertices. Let $u$ and $v$ be two distinct vertices of $T$. We show that in $\bar{T}$ there always exists a path connecting $u$ and $v$.

**Case 1.** Vertices $u$ and $v$ are not adjacent in $T$. Then these vertices are adjacent in $\bar{T}$ and are thus connected by an edge. We are done.

**Case 2.** Vertices $u$ and $v$ are adjacent in $T$. Then either

\begin{align*}
    &d(u) = 1 \text{ and } d(v) = 1 \; \text{(subcase 2.1)}, \; \text{or} \\
    &d(u) = 1 \text{ and } d(v) > 1 \; \text{(subcase 2.2)}, \; \text{or} \\
    &d(u) > 1 \text{ and } d(v) = 1 \; \text{(subcase 2.3)}, \; \text{or} \\
    &d(u) > 1 \text{ and } d(v) > 1 \; \text{(subcase 2.4)}. \\
\end{align*}

**Subcase 2.1** implies $T = S_2$, contradiction.

**Subcase 2.2.** If $d(u) = 1$ and $d(v) > 1$ then either (i) all neighbors of $v$ are of degree 1, or (ii) at least one neighbor of $v$, say $x$, is of degree greater than one. If (i) holds, then $T$ is a star, contradiction. It (ii) holds, then $x$ has a further neighbor $y$. The vertex $y$ differs from $u$, and $y$ is not adjacent to either $v$ or $u$, because otherwise $T$ would possess a cycle. Then in $\bar{T}$, $u$ and $y$ are adjacent and $v$ and $y$ are adjacent. Therefore, in $\bar{T}$ there is a path $(u, y, v)$ connecting $u$ and $v$.

**Subcase 2.3** is treated in a fully analogous manner.

**Subcase 2.4.** The vertex $u$ has a neighbor $x$ and the vertex $v$ has a neighbor $y$. The vertices $x$ and $y$ are different and not adjacent, because otherwise $T$ would possess a cycle. Then in $\bar{T}$, $u$ is adjacent to $y$, $v$ is adjacent to $x$, and $x$ is adjacent to $y$. Consequently, in $\bar{T}$, the vertices $u$ and $v$ are joined by the path $(u, y, x, v)$.

By this all possibilities have been exhausted, and the general validity of Lemma 7 has been verified.

Combining Lemma 7 with Lemma 5 we conclude:
Lemma 8. If $T$ is any $n$-vertex tree, not isomorphic to $S_n$, then $\mu_1(T) < n$.

Combining Lemmas 6 and 8 we reach our main result:

Theorem 1. Among all trees with a fixed number of vertices the star has the greatest value of the greatest Laplacian eigenvalue.

Although elementary, the result of Theorem 1 seems not to be previously reported in the mathematical literature. Even worse, in a recent paper [1] the property $\mu_1 = n$ was erroneously attributed only to the complete graph, from which it would (erroneously) follow that the greatest value of the greatest Laplacian eigenvalue of any $n$-vertex tree is less than $n$.

References


