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SOME PROBLEMS ABOUT THE LIMIT OF A REAL–VALUED FUNCTION

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1. In [1] S. Banach solved the problem of the existence of a (non-unique) linear shift-invariant functional on the space of all bounded functions defined on the semi-axis $t \geq 0$.

2. Let now a be sufficiently large (written $a > a_0$ for some a_0). Denote by Ω the real vector space of all real-valued functions on $[0, \infty)$ and bounded on $[a, \infty)$. This paper is organized as follows. First we will show the existence of a family of functionals on the space Ω containing Banach shift-invariant functionals. Next, by these functionals we shall define the limit of $f(t)$ as $t \to \infty$, $f \in \Omega$, and show that this definition is equivalent to the classical definition of this limit. Further, we show some theorems characterizing the limit of a function $f(t), t \geq 0$ as $t \to +\infty$. Each of these theorems gives an answer to the question what (new) conditions must satisfy a function $f \in \Omega$ such that the limit of $f(t)$ as $t \to +\infty$ exists.

Again, let a be sufficiently large. Define the functional p on the space Ω by

$$p(f) = \sup_{n,\xi_k} \left\{ \limsup_{t\to\infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} f(t+\xi_k) \right| \right\} \quad (f \in \Omega), \tag{1}$$

where the supremum is taken over all n and ξ_k ($\xi_k \geq 0,\ n = 1, 2, \ldots$). The functional p is seen to be real-valued and it clearly satisfies the conditions

$$p(f) \geq 0,\ p(af) = |a|p(f),\ p(f+g) \leq p(f) + p(g), \quad (a \in \mathcal{R};\ f, g \in \Omega);$$

that is, p is a symmetric convex functional on the space Ω. According to a corollary of Hahn-Banach theorem (see also [2], Exercise 11.2, p.187) there exists a nontrivial linear functional L on the space Ω such that

$$|L(f)| \leq p(f), \quad f \in \Omega. \tag{2}$$

We next wish to show that the functional L satisfying the above conditions is not unique. To do this, let Ω_0 be the space of all functions $f \in \Omega$ having $\lim_{t\to\infty}$. Then clearly we have

$$p(f) = L(f) = 0, \quad f \in \Omega_0. \tag{3}$$

Take now a number $s \in \mathcal{R}$ ($s \neq 0$) and define the function g by $g(t) = s,\ t \geq 0$. Then $g \in \Omega \setminus \Omega_0$ and $p(g) = |s| > 0$. To extend the functional $L : \Omega_0 \mapsto \mathcal{R}$ to the space spanned by Ω_0 and $\{g\}$ (that is, the space $\Omega_0 \cup \{g\}$), the value $L(g)$ we can choose arbitrarily in the segment

$$[-p(g), p(g)].$$

Thus, it is possible to extend the functional L such that it has distinct values at the point $g \in \Omega$. In other words, there are functionals on the space Ω satisfying the above conditions with distinct values at point g; that is, the functional satisfying the above conditions is not unique. Indeed, we can take the value $L(g)$ arbitrarily in the segment $[k, K]$, where

$$k = \sup_{f\in\Omega_0} \{-p(f+g)\}, \qquad K = \inf_{f\in\Omega_0} \{p(f+g)\}$$

since $L(f) = 0$, $\forall f \in \Omega_0$ (see, for example, [4], p. 222). Further, by (1), we have $p(f+g) = p(g)$ since $f(t) + g(t) \to s$, as $t \to +\infty$. Consequently, we can take arbitrarily $L(g)$ in $[-p(g), p(g)]$.

Now, we show the following lemma.

Lemma. *Let X be a real linear space and $p : X \mapsto \mathcal{R}$ a functional satisfying the following conditions*

$$q(x) \geq 0, \; p(ax) = |a|p(x), \; p(x+y) \leq p(x) + p(y) \quad (a \in \mathcal{R}; \; x, y \in X).$$

Then, for each $x_0 \in X$, there exists a linear functional L on X such that

$$(\forall x \in X) \; |L(x)| \leq p(x), \quad L(x_0) = p(x_0).$$

Proof. The set $X_0 = \{\alpha x_0 : \; \alpha \in \mathcal{R}\}$ clearly is a subspace of the space X, and L_0, defined by

$$L_0(\alpha x_0) = \alpha p(x_0) \quad (\alpha \in \mathcal{R}),$$

is a linear functional on the subspace X_0 satisfying the condition

$$|L_0(\alpha x_0)| = |\alpha p(x_0)| = |\alpha| p(x_0) = p(\alpha x_0) \quad (\alpha \in \mathcal{R}).$$

By a version of Hahn-Banach theorem (see [3], theorem 11.2, p. 181) there exists a linear functional L on the space X extending L_0 and satisfying the conditions

$$(\forall x \in X) \; |L(x)| \leq p(x)$$

and

$$L(x_0) = L_0(x_0) = 1 \cdot p(x_0) = p(x_0).$$

Denoting now by Π the family of the functionals L obtained before, then for each $s \in \mathcal{R}$ we have

$$(\forall L \in \Pi) L(f - s) = 0 \quad \text{iff} \quad p(f - s) = 0 \quad (f \in \Omega).$$

Indeed, $p(f-s)=0$ clearly implies $L(f-s)=0,\ \forall L\in\Pi$. Also, the implication

$$(\forall L\in\Pi)L(f-s)=0\ \Rightarrow\ p(f-s)=0$$

is equivalent to the implication

$$p(f-s)>0\ \Rightarrow\ (\exists L\in\Pi)L(f-s)\neq 0$$

which, by the lemma proved before, is valid. So, (4) is true.

We now summarize the results obtained before as the following statement.

Theorem 1. *There exists the family* Π *of functionals* L *defined on the space* Ω *such that, for all* $a,b\in\mathcal{R},\ f,g\in\Omega,\ s\in\mathcal{R}$, *we have*

1^0 $\quad L(af+bg)=aL(f)+bL(g)$,

2^0 $\quad |L(f)|\le p(f)$,

3^0 $\quad (\forall L\in\Pi)L(f-s)=0$ iff $p(f-s)=0$.

THE LIMIT OF A REAL-VALUED FUNCTION

In [3] was defined the almost convergence of a sequence by Banach shift-invariant functionals. Analogously, we here define the limit of a function $f\in\Omega$ by the functionals from the theorem 1.

Definition 1. *Let* $f\in\Omega$. *We will say that* $f(t)$ *has limit* s *as* $t\to+\infty$ *(written, as usual,* $\lim_{t\to+\infty}=s$ *or* $f(t)\to s$ *as* $t\to+\infty$*) if*

$$(\forall L\in\Pi)L(f-s)=0. \tag{5}$$

We now shall show that the limit of a function $f \in \Omega$ defined in such way is uniquely determined. Indeed, for any two limits s' and s'' of a function f, define the functions g and h by

$$g(t) = s' \quad \text{and} \quad h(t) = s'' \ (t \geq 0).$$

Then, by (5), we have

$$(\forall L \in \Pi) L(h - g) = L(f - g) - L(f - h) = L(f - s') - L(f - s'') = 0$$

which, by (4) and (1), implies

$$p(h - g) = |s'' - s'| = 0 \quad \text{or} \quad s' = s''.$$

Now is arised the question is the definition 1. equivalent to the corresponding classical definition. A positive answer gives the following statement.

Theorem 2. *The definition 1 and the corresponding classical definition are equivalent.*

Proof. Let $\lim_{t \to +\infty} f(t) = s$ in the sense of definition 1. Then, by (5), (4) and (1), for all n $(= 1, 2, \ldots)$ and $\xi_k \geq 0$, we have

$$\limsup_{t \to \infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} [f(t + \xi_k) - s] \right| = 0.$$

Thence for $n = 1$ and $\xi_0 = 0$

$$\limsup_{t \to +\infty} |f(t) - s| = 0 \quad \text{or} \quad f(t) \to s \quad \text{as} \quad t \to +\infty$$

in the sense of the corresponding classical definition.

Conversely, let $\lim_{t \to +\infty} f(t) = s$ in the sense of the classical definition. Then, for all n $(= 1, 2, \ldots)$ and $\xi_k \geq 0$ $(k = 0, 1, 2, \ldots)$ we have

$$\limsup_{t \to \infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} [f(t + \xi_k) - s] \right| = \lim_{t \to \infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} [f(t + \xi_k) - s] \right| = 0.$$

Hence

$$p(f - s) = \sup_{n, \xi_k} \left\{ \limsup_{t \to \infty} \frac{1}{n} \left| \sum_{k=0}^{n-1} [f(t + \xi_k) - s] \right| \right\} = 0.$$

This, by (4), implies (5); so, $f(t) \to s$ as $t \to +\infty$ in the sense of definition 1. which completes the proof.

Notice. It is clear from expressions and results obtained before (up to theorem 3., inclusive), as in S. Banach, G. G. Lorentz and other's papers, that it is possible to obtain not only a generalization of usual limit of a function $f(t)$, $t \geq 0$ as $t \to +\infty$, but usual limit of $f(t)$, $t \geq 0$ as $t \to +\infty$ itself, too.

We can now proceed to the following three statements whose sense is to additionally characterize the limit of a function $f(t),\ t \geq 0$ by distinct conditions and expressions which analogous to the expressions obtained before.

Theorem 3. *For each function $f(t),\ t \geq 0$ we have $f(t) \to s$ as $t \to +\infty$ iff*

$$\frac{1}{n}\sum_{k=0}^{n-1} f(t+\xi_k) \to s \quad \text{as} \quad t \to +\infty \tag{6}$$

uniformly in n and ξ_k ($\xi_k \geq 0,\ n = 1, 2, \ldots$).

Proof. Suppose the condition (6) is true. Then for $n = 1$ and $\xi_0 = 0$ we have $f(t) \to s$ as $t \to +\infty$. Conversely, let $f(t) \to s$ as $t \to +\infty$. Then for any $\varepsilon > 0$ there exists a number t_0 such that for all $\xi_k \geq 0$ we have

$$|f(t+\xi_k) - s| < \varepsilon, \quad t > t_0.$$

Now, for all n $(= 1, 2, \ldots)$ and $\xi_k \geq 0$ $(k = 0, 1, 2, \ldots)$ we have

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} f(t+\xi_k) - s\right| \leq \frac{1}{n}\sum_{k=0}^{n-1} |f(t+\xi_k) - s| < \varepsilon, \quad t > t_0.$$

Since $\varepsilon > 0$ is arbitrary, this means that

$$\frac{1}{n}\sum_{k=0}^{n-1} f(t+\xi_k) \to s \quad \text{as} \quad t \to +\infty$$

uniformly in n and $\xi_k \geq 0$ which completes the proof

The following interesting theorem is a modification of the theorem 3.

Theorem 4. *For each function* $f(t),\ t \ge 0$ *we have* $f(t) \to s$ *as* $t \to +\infty$ *iff*

$$\frac{1}{n}\sum_{k=in}^{(i+1)n-1} f(t+\xi_k) \to s \quad as \quad t \to +\infty \tag{7}$$

uniformly in n, i *and* ξ_k $(\xi_k \ge 0,\ k = 0, 1, 2, \ldots, n;\ i = 0, 1, 2, \ldots)$.

Proof. Suppose the condition (7) is true. Then for $i = 0$ (7) implies (6). Hence, by the theorem 3, the condition (7) is sufficient.

Conversely, let $f(t) \to s$ as $t \to +\infty$. Then for each $\varepsilon > 0$ there exists a number t_0 such that

$$|f(t) - s| < \varepsilon, \quad t > t_0$$

which, for all n, i and $\xi_k \ge 0$, implies

$$|f(t + \xi_k + in) - s| < \varepsilon, \quad t > t_0.$$

Hence, for all n, i and $\xi_k \ge 0$, we have

$$\left|\frac{1}{n}\sum_{k=in}^{(i+1)n-1} f(t+\xi_k) - s\right| \le \frac{1}{n}\sum_{k=0}^{n-1}|f(t+\xi_k+in) - s| < \varepsilon,\ t > t_0.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\frac{1}{n}\sum_{k=in}^{(i+1)n-1} f(t+\xi_k) \to s \quad \text{as} \quad t \to +\infty$$

uniformly in n, i and $\xi_k \ge 0$ $(k = 0, 1, 2, \ldots;\ n = 1, 2, \ldots;\ i = 0, 1, 2, \ldots)$ which completes the proof.

Now, we will show the following applicable theorem containing a new restrictive condition.

Theorem 5. *Let* $f \in \Omega$ *be a continuous function on* $[a, +\infty)$*,* $a > a_0$*. Then* $f(t) \to s$ *as* $t \to +\infty$ *iff*

$$\frac{1}{T}\int_a^{a+T} f(t)\,dt \to s \quad itas \quad a \to +\infty$$

uniformly in T (> 0).

Proof. Suppose $f(t) \to s$ as $t \to +\infty$. Then for each $\varepsilon > 0$ there exists a number t_0 $(> a_0)$ such that

$$|f(t) - s| < \varepsilon,\ t > t_0.$$

Hence, for all T (> 0) and all $a > t_0$, we have

$$\left|\frac{1}{T}\int_a^{a+T} f(t)\,dt - s\right| \le \frac{1}{T}\int_a^{a+T} |f(t) - s|\,dt < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this means that

$$\frac{1}{T}\int_a^{a+T} f(t)\,dt \to s \quad \text{as} \quad a \to +\infty$$

uniformly in T (> 0), so the condition (8) is necessary.

Conversely, it is clear that the proof (8) implies $f(t) \to s$ as $t \to +\infty$ can be reduced to the case $s = 0$. Accordingly, let us suppose that

$$\frac{1}{T}\int_a^{a+T} f(t)\,dt \to 0 \quad \text{as} \quad a \to \infty \tag{9}$$

uniformly in T (> 0) holds and that

$$f(t) \to 0 \quad \text{as} \quad t \to +\infty \tag{10}$$

does not. Then we have

$$\lambda = \limsup_{t\to+\infty} f(t) > 0 \quad \text{or} \quad \liminf_{t\to+\infty} f(t) < 0. \tag{11}$$

We will consider only the first case in (11), because the second case is simply reducible to the first one. From (10) follows the existence of an $a_0 > 0$ such that we have

$$\frac{1}{T}\int_a^{a+T} f(t)\,dt < \frac{1}{2}\lambda, \quad \text{for} \quad a > a_0,\ T > 0, \tag{12}$$

and from (11) follows the existence of a $b > a_0$ such that

$$f(b) > \frac{1}{2}\lambda. \tag{13}$$

Then we have

$$\frac{1}{T}\int_a^{b+T} f(t)\,dt < \frac{1}{2}\lambda \quad (T > 0)$$

and hence, on account of the continuity of f at the point b and by a known theorem,

$$f(b) = \lim_{T\to+0} \frac{1}{T}\int_a^{b+T} f(t)\,dt \le \frac{1}{2}\lambda$$

contrary to (13). This contradiction proves our assertion and completes the proof.

References

[1] S. Banach, *Theorie des operations linearies,* Warszawa (1932), 28–29.

[2] G. Bachman and L. Narici, *Functional Analysis,* Academic Press, New York and London (1966), p. 181.

[3] D. Hajduković, *The functionals of the kind of Banach limits,* Publications de L'Institut Mathematique **19** (33) (1975).

[4] S. Aljančić, *Uvod u realnu i funkcionalnu analizu,* Građevinska knjiga, Beograd (1968), pp. 222–224.