SOME GENERALIZATION OF WEIGHTED NORM INEQUALITIES FOR CERTAIN CLASS OF INTEGRAL OPERATORS

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Abstract. A generalization is obtained for a non-negative weight function $w$ for which there is a non-negative weight function $\nu < \infty \mu$-almost everywhere such that $T$ maps $L^p(\nu)$ to $L^q(w)$, i.e.

$$\left[ \int_X (Tf)^q w \, d\mu \right]^{1/q} \leq C \left[ \int_X f^p \nu \, d\mu \right]^{1/p} \quad \text{for all } f \geq 0 \quad (1.1)$$

and $C$ is a constant depending on $K, p, q$ but independent of $f$. Furthermore, for $T$ sublinear operator generalization is obtained for weight functions for which $T$ is bounded from $L^q(R^n, \omega \, dx)$ to $L^p(R^n, \nu \, dx)$ for some nontrivial $w$.

1. INTRODUCTION

Let $(X, A, \mu)$ be a $\sigma$-finite measure space and let $K(x, y)$ be a non-negative and measurable on $X \times X$. Set $Tf(x) = \int_X K(x, y)f(y) \, dy$ and it’s dual $T^*f(y) = \int_X K(y, x)f(y) \, dy$ for non-negative function $f$.

For $1 < p < \infty$, we shall consider the weighted norm inequality

$$\int_X (Tf)^p w \, d\mu \leq C \int_X f^p \nu \, d\mu \quad \text{for all } f \geq 0, \quad (1.2)$$
where \( w \) and \( \nu \) are non-negative measurable weight functions on \( X \).

In [4], R. Kerman and E. Sawyer proved the following theorem on weighted norm inequalities for positive linear operators.

**Theorem 1.1.** Let \( 1 < p < \infty \) and suppose \( w \) is a weight on \( X \). Then there is a weight \( \nu \), finite \( \mu \)-almost everywhere on \( X \), such that the weighted norm inequality (1.2) holds if and only if there exists a positive function \( \Phi \) on \( X \) with

\[
\int_X (T\Phi)^p w \, d\mu < \infty \quad \text{or equivalently} \quad \Phi^{1-p} T^* \big( (T\Phi)^{p-1} w \big) < \infty \quad \mu-\text{almost everywhere}. \tag{1.3}
\]

This theorem is known to have extended some earlier results of B. Muchenhoupt in [5]. The main objective of the present paper is to prove a result which is more general than Theorem 1.1.

Throughout this paper, \( p' \) denotes the conjugate index of \( p \), \( p \neq 0 \) and is defined by \( \frac{1}{p} + \frac{1}{p'} = 1 \) if \( p = 1 \), the conjugate of \( q \) is defined in the same way.

2. MAIN RESULTS

We state our main result.

**Theorem 2.1.** Let \( 1 < p < \infty \) and suppose \( w \) is a weight on \( X \). Define the sublinear operator \( T \) by

\[
T(f + g)(x) = \int_X K(x, y)(f + g)(y) \, d\mu(y). \tag{2.1}
\]

Then, there exists a weight function \( \nu \), finite \( \mu \)-almost everywhere on \( X \) such that

\[
\int_X \{T(f + g)\}^p w \, d\mu \leq C(K, p) \int_X (f^p + g^p) \nu \, d\mu \tag{2.2}
\]

holds, for all \( f, g > 0 \), if and only if there is a positive function \( \Phi \) and \( \theta \) on \( X \) with

\[
\int_X (T\Phi)^p w \, d\mu < \infty \quad \text{and} \quad \int_X (\Phi^{1-p} T^* \big( (T\Phi)^{p-1} w \big) \, d\mu < \infty \quad \mu-\text{almost everywhere}. \tag{2.3a}
\]
\[ \int_X (T \theta)^p w \, d\mu < \infty \quad \text{or equivalently} \quad (2.3b) \]
\[ \Phi^{1-p} T^* ((T \Phi)^{p-1} w) < \infty \quad \text{and} \quad (2.4a) \]
\[ \theta^{1-p} T^* ((T \theta)^{p-1} w) < \infty \quad \mu - \text{almost everywhere}, \quad \text{and} \quad (2.4b) \]
\[ C(K, p) = \max\{C_1(K, p), C_2(K, p)\} \]
is a constant independent of \( f \) and \( g \).

Indeed, the weighted inequality (2.2) holds with \( \nu_1 \) and \( \nu_2 \) equal to the weight in (2.4a) and (2.4b) respectively.

**Proof.** Let
\[ I = \int_X (T(f + g))^p w \, d\mu. \]

Then
\[
I = \int_X \left\{ \int_X K(x, y)(f + g) \, d\mu(y) \right\}^p w \, d\mu
\]
\[
= \int_X \left\{ \int_X (K(x, y)f(y) + K(x, y)g(y)) \, d\mu(y) \right\}^p w \, d\mu
\]
\[
\leq \int_X \left\{ \int_X K(x, y)f(y) \, d\mu(y) \right\}^p w \, d\mu + \int_X \left\{ \int_X K(x, y)g(y) \, d\mu(y) \right\}^p w \, d\mu
\]
by Minkowski’s inequality
\[
\leq \int_X \left( \int K(x, y)f(y)^{p\Phi^{p/p'}} \, d\mu(y) \right) \left( \int K(x, y)\Phi \, d\mu(y) \right)^{p/p'} w \, d\mu
\]
\[
+ \int_X \left( \int K(x, y)g(y)^{p\theta^{p/p'}} \, d\mu(y) \right) \left( \int K(x, y)\theta \, d\mu(y) \right)^{p/p'} w \, d\mu
\]
by Holder’s inequality
\[
= \int_X \left\{ \int K(x, y)f(y)^p \Phi^{1-p} \, d\mu(y) \right\} \left( \int K(x, y)\Phi \, d\mu(y) \right)^{p-1} w \, d\mu
\]
\[
+ \int_X \left\{ \int K(x, y)g(y)^p \theta^{1-p} \, d\mu(y) \right\} \left( \int K(x, y)\theta \, d\mu(y) \right)^{p-1} w \, d\mu
\]
\[
= \int_X [(T f^p \Phi^{1-p})(T \Phi)^{p-1} w] \, d\mu + \int_X [(T g^p \theta^{1-p})(T \theta)^{p-1} w] \, d\mu
\]
\[
\int_X f^p \Phi^{1-p} (T\Phi)^{p-1} w \, d\mu + \int_X g^p \theta^{1-p} (T\theta)^{p-1} w \, d\mu \\
\leq C_1(K, p) \int_X f^p \nu_1 \, d\mu + C_2(K, p) \int_X g^p \nu_2 \, d\mu \\
= C(K, p) \int_X (f^p + g^p) \nu \, d\mu ,
\]
where \( \nu = \max\{\nu_1, \nu_2\} \) and \( C(K, p) = \max\{C_1, C_2\} \) which yields (2.2) with \( \nu \) equal to the weight in (2.4a) and (2.4b).

Conversely, assume (2.2) holds for some \( \nu < \infty \) \( \mu \)-almost everywhere. Using the \( \sigma \)-finiteness of \( \mu \). One can easily construct a positive functions \( \Phi \) and \( \theta \) such that

\[
\int_X (\Phi^p + \theta^p) \nu \, d\mu < \infty
\]

and hence such that (2.3) holds. Finally, suppose (2.3) holds and let \( \nu \) denotes the weight in (2.4a) and (2.4b). Then

\[
\int_X (\Phi^p + \theta^p) \nu \, d\mu = \int_X \Phi^p \nu_1 \, d\mu + \theta^p \nu_2 \, d\mu \\
= \int_X \Phi^p \left( \Phi^{1-p} T^* \left[ (T\Phi)^{p-1} w \right] \right) \, d\mu + \int_X \theta^p \left( \theta^{1-p} T^* \left[ (T\theta)^{p-1} w \right] \right) \, d\mu \\
= \int_X T\Phi (T\Phi)^{p-1} w \, d\mu + \int_X T\theta (T\theta)^{p-1} w \, d\mu \\
= \int_X (T\Phi)^p w \, d\mu + \int_X (T\theta)^p w \, d\mu \\
= \int_X (T+\theta)^p w \, d\mu < \infty \quad \text{by (2.3)}.
\]

Since \( \Phi > 0, \theta > 0 \), we conclude \( \nu < \infty, \mu \)-almost everywhere and this completes the proof of the Theorem.

**Remark 2.1.** When \( g \equiv 0 \) on \( X \), Theorem (2.1) reduces to Kerman and Sawyer result [4].

**Theorem 2.2.** Let \( 1 < p < \infty \) and suppose \( w \) is a weight on \( X \). Define the sublinear operator \( T^* \) by

\[
T^*(f + g)(x) = \int_X K(y, x)(f + g)(y) \, d\mu(y).
\]
Then, there exists a weight function $\nu$, finite $\mu$-almost everywhere on $X$ such that

$$
\int_X \{T^*(f + g)\}^p w \, d\mu \leq C(K, p) \int_X (f^p + g^p) \nu \, d\mu
$$  \hspace{1cm} (2.7)

holds, for all $f, g > 0$, if and only if there is a positive function $\Phi$ and $\theta$ on $X$ with

$$
\int_X (T^*\Phi)^p w \, d\mu < \infty \hspace{1cm} (2.8a)
$$

$$
\int_X (T^*\theta)^p w \, d\mu < \infty \hspace{1cm} \text{or equivalently} \hspace{1cm} (2.8b)
$$

$$
\Phi^{1-p} T \left( (T^*\Phi)^{p-1} w \right) < \infty \hspace{1cm} \text{and} \hspace{1cm} (2.9a)
$$

$$
\theta^{1-p} T \left( (T^*\theta)^{p-1} w \right) < \infty \hspace{1cm} \mu-\text{almost everywhere}, \hspace{1cm} (2.9b)
$$

$$
C(K, p) = \max \{C_1(K, p), C_2(K, p)\} \hspace{1cm} (2.10)
$$

is a constant independent of $f$ and $g$.

Indeed, the weighted inequality (2.7) holds with $\nu_1$ and $\nu_2$ equal to the weight in (2.9a) and (2.9b) respectively.

**Proof.** The proof is immediate from the proof of Theorem 2.1. by defining $T^*$ as

$$(T^*f)(x) = \int_X K(y, x) f(y) \, dy.$$

3. THE CASE $1 < p \leq q \leq \infty$

In this section, we shall obtain some weighted norm inequalities for mixed norm under more restricted condition on $\nu$ and $w$. See [1], [2], and [3] for related work.

**Theorem 3.1.** Let $1 < p \leq q = \infty$ and suppose $u = w^{1/q}$ is a weight on $X$, then there is a weight $\nu$, finite $\mu$-almost everywhere on $X$ such that the weighted norm inequality:

$$
\left[ \int_X (Tf)^q w \, d\mu \right]^{1/q} \leq C \left[ \int_X f^p \nu \, d\mu \right]^{1/p}
$$  \hspace{1cm} (3.1)

holds, if and only if there is a positive function $\Phi$ on $X$ satisfying

$$
\Phi(y)^p \leq \nu
$$  \hspace{1cm} (3.2)
and $C = C(K, p, q)$ is a constant independent of $f$.

**Proof.** Let

$$I = \left\{ \int_X (Tf)^q w \, d\mu \right\}^{1/q}$$

Then

$$I = \left\{ \int_X (uTf)^q d\mu \right\}^{1/q} \leq \sup_{x < \infty} \left\{ u(x) \int_X K(x, y) f(y) \, dy \right\}$$

$$= \sup_{x < \infty} \left\{ u(x) \int_X K(x, y) f(y) K(x, y)^{1-\beta} \, dy \right\}$$

$$\leq \sup_{z < x} \text{ess} K(x, z) u(x) \int_X K(x, y)^{1-\beta} f(y) \, dy$$

$$= \sup_{z < x} \text{ess} K(x, z) u(x) \int_X K(x, y)^{1-\beta} \Phi(y)^{-1} f(y) \, dy$$

$$\leq \sup_{z < x} \text{ess} K(x, z) u(x) \left\{ \int_X K(x, y)^{(1-\beta)p'} \Phi(y)^{-p'} \, dy \right\}^{1/p'} \left\{ \int f(y)^p \Phi(y)^p \, dy \right\}^{1/p}$$

by Holder’s inequality.

The integral

$$\left\{ \int_X K(x, y)^{(1-\beta)p'} \Phi(y)^{-p'} \, dy \right\}^{1/p'} \leq C \left\{ \sup_{t > x} \text{ess} K(t, x)^{\beta} u(t) \right\}^{-1}$$

since $u(x)$ and $\Phi(x)$ depend on $p$ and $q$ with constant $C$.

Hence,

$$I \leq C \sup_{z < x} K(x, z)^{\beta} u(x) \left\{ \sup_{t > x} K(t, x)^{\beta} u(t) \right\}^{-1} \left\{ \int f(y)^p \Phi(y)^p \, dy \right\}^{1/p}.$$

Now

$$\sup_{z < x} K(x, z)^{\beta} u(x) \left\{ \sup_{t > x} K(t, x)^{\beta} u(t) \right\}^{-1} \leq \sup_{z < x} K(x, z)^{\beta} u(t) \left\{ \sup_{z > x} K(t, z)^{\beta} u(t) \right\}^{-1} = 1$$

since $K(t, \cdot)$ is not-decreasing.

Therefore,

$$I = C \left( \int_X f(y)^p \Phi(y)^p \, dy \right)^{1/p} \leq C \left\{ \int f(y)^p \nu \, d\mu(y) \right\}^{1/p}.$$
This completes the proof.

**Remark 3.1.** If \( q = p \), the above Theorem reduces to the result obtained by Kerman and Sawyer [4].

**Theorem 3.2.** Let \( 1 < p \leq q = \infty \) and suppose \( u = w^{1/q} \) is a weight on \( X \). Then there is a weight \( \nu \), finite \( \mu \)-almost everywhere on \( X \) such that the weighted norm inequality:

\[
\left[ \int_X (T^* f)^q w \, d\mu \right]^{1/q} \leq C \left[ \int_X f^p \nu \, d\mu \right]^{1/p} \quad \text{for all } f \geq 0,
\]

(3.3) holds, if and only if there is a positive function \( \Phi \) on \( X \) satisfying

\[
\Phi(x)^p \leq \nu,
\]

(3.4) and \( C = C(K, p, q) \) is a constant independent of \( f \).

**Proof.** Follows directly from the prove of Theorem 3.1. by defining \( T^* \) as \((T^* f)(x) = \int_X K(y, x) f(y) \, dy\).

**Theorem 3.3.** Let \( 1 < p \leq q < \infty \) and suppose \( u = w^{1/q} \) is a weight on \( X \). Then there is a weight \( \nu \), finite \( \mu \)-almost everywhere on \( X \) such that the weighted norm inequality:

\[
\left[ \int_X (T f)^q w \, d\mu \right]^{1/q} \leq C \left[ \int_X f^p \nu \, d\mu \right]^{1/p} \quad \text{for all } f \geq 0,
\]

(3.5) holds, if and only if there is a positive function \( \Phi \) on \( X \) satisfying

\[
\Phi(x)^p \leq \nu,
\]

(3.6) with

\[
s(x) \leq \left( \int_X K(y, z) \Phi(z)^{-p} \, dz \right)^{1/(p+1)},
\]

and \( C = C(K, p, q) \) is a constant independent of \( f \).

**Proof.** Let

\[
I = \int_X [T f]^q w \, d\mu(x).
\]
Then
\[ I = \int_X [u(x)Tf]^q d\mu(x) = \int_X \left[ u(x) \int_X K(x, y)f(y) d\mu(y) \right]^q d\mu(x) \]
\[ = \int_X u(x)^q \left[ \left( \int_X K(x, y)^\beta f(y)\Phi(y)s(y)K(x, y)^{1-\beta}\Phi(y)^{-1}s(y)^{-1} \right) d\mu(y) \right]^q d\mu(x) \]
\[ \leq \int_X u(x)^q \left[ \left( \int_X K(x, y)^{\beta p} (f(y)\Phi(y)s(y))^p d\mu(y) \right)^{q/p} \right. \]
\[ \times \left( \int_X K(x, y)^{(1-\beta)q/p} \Phi(y)^{-q/p} s(y)^{-q/p} d\mu(y) \right)^{q/p} d\mu(x) \]
by Holder’s inequality.
\[ = \int_X u(x)^q \left[ \left( \int_X K(x, y) (f(y)\Phi(y)s(y))^p d\mu(y) \right)^{q/p} \right. \]
\[ \times \left( \int_X K(x, y)^{-q/p} \Phi(y)^{-q/p} s(y)^{-q/p} d\mu(y) \right)^{q/p} d\mu(x) \]
\[ \leq (p' + 1)^{q/p'} \int_X \left[ u(x)^q \left( \int_X K(x, y) (f(y)\Phi(y)s(y))^p d\mu(y) \right)^{q/p} s(x)^{q/p'} \right] d\mu(x) \]
by definition of \( s(x) \)
\[ \leq (p' + 1)^{q/p'} \left\{ \int_X \left( \int_X K(x, y)u(x)^q s(x)^{q/p'} d\mu(x) \right)^{p/q} (f(y)\Phi(y)s(y))^p d\mu(y) \right\}^{q/p} \]
by Minkowski’s integral inequality.
\[ = (p' + 1)^{q/p'} \left\{ \int_X \left( \int_X K(x, y)u(x)^q \left( K(y, z)\Phi(z)^{-q/p'} dz \right)^{q/p'} d\mu(x) \right)^{p/q} \right\}^{q/p} \]
\[ \times (f(y)\Phi(y)s(y))^p d\mu(y) \right\}^{q/p} \]
But,
\[ \int_X K(x, y)\Phi(x)^{-q/p'} dx \leq C^{p'} \left\{ \int_X K(z, y)u(z)^q dz \right\}^{-q/p'} . \]
Since \( u(x) \) and \( \Phi(x) \) depend on \( p \) and \( q \) with constant \( C \)
\[ \leq (p' + 1)^{q/p'} C^{q/(p'+1)} \left\{ \int_X \left( \int_X K(x, y)u(x)^q \left( \int_X K(z, y)u(z)^q dz \right)^{-1/(p'+1)} d\mu(x) \right)^{p/q} \right\}^{p/q} \]
\[ \times (f(y)\Phi(y)s(y))^p d\mu(y) \right\}^{q/p} \]
\[ \leq (p' + 1)^{q/p'} C^{q/(p'+1)} \left( \frac{p' + 1}{p'} \right)^{p/q} \left\{ \int_X \left( \int_X K(z, y)u(z)^q dz \right)^{p/q/(q(p'+1))} \right\} \]
\[ \times (f(y)\Phi(y)s(y))^p d\mu(y) \right\}^{q/p} . \]
But, since $u(x)$ and $\Phi(x)$ depend on $p$ and $q$ with constant $C$

$$\leq (p' + 1)^{q/p'} C^{q/(q' + 1)} \left( \frac{p' + 1}{p'} \right)^{p/q} \left( \int_X \left( \int_X K(y, z) \Phi(z)^{-p} d z \right)^{-(p' + 1)} \right) \times \left( f(y) \Phi(y) s(y) \right)^{q/p}$$

$$= (p' + 1)^{q/p'} C^{q/(q' + 1)} \left( \frac{p' + 1}{p'} \right)^{p/q} C^{q/(q' + 1)} \left( \int_X s(y)^{-p} s(y)^p f(y)^p \Phi(y)^p d \mu(y) \right)^{q/p}.$$

Therefore,

$$I^{1/q} = (p' + 1)^{q/p' - p/q} \left( \int_X f(y)^p \Phi(y)^p d \mu(y) \right)^{1/p} \leq (p' + 1)^{q/p' - p/q} \left( \int_X f(y)^p \nu^p d \mu(y) \right)^{1/p}.$$

This completes the proof.

**Theorem 3.4.** Let $1 < p \leq q < \infty$ and suppose $u = w^{1/q}$ is a weight on $X$. Then there is a weight $\nu$, finite $\mu$-almost everywhere on $X$ such that the weighted norm inequality:

$$\left[ \int_X (T^* f)^q w d \mu \right]^{1/q} \leq C \left[ \int_X f^p \nu^p d \mu \right]^{1/p}$$

for all $f \geq 0$, (3.7) holds, if and only if there is a positive function $\Phi$ on $X$ satisfying:

$$\Phi(x)^p \leq \nu$$

with $s(x) \left( \int_X K(y, z) \Phi(z)^{-p} d z \right)^{1/(p + 1)}$ and $C = C(K, p, q)$ is a constant independent of $f$.

**Proof.** Follows directly from the proof of Theorem 3.2. by defining $T^*$ as

$$(T^* f)(x) = \int_X K(y, x) f(y) d y.$$

**Remark 3.2.** If we put $q = p$ we obtain Kerman and Sawyer result [4]. Hence, our result gives a better bound than Theorem 1.1.
4. CONSEQUENCES OF OUR MAIN RESULTS

**Corollary 4.1.** Suppose that \( \Phi, w \geq 0 \) are locally integrable with respect to Lebesgue measure on \( \mathbb{R}^n \) and that \( \Phi(x) = \Phi(|x|) \) is non-increasing as a function of \( |x| \).

Define the convolution operator \( T \) by

\[
(Tf)(x) = (\Phi^*f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y) \, dy
\]

for a fixed \( p \in (1, \infty) \) and a constant \( C \), depending on \( p \) and \( q \). Then there exists \( \nu(x) < \infty \) almost everywhere and \( C > 0 \) such that

\[
\left[ \int_{\mathbb{R}^n} (Tf)^q w \, dx \right]^{1/q} \leq \left[ C \int_{\mathbb{R}^n} f^p \nu \, dx \right]^{1/p}
\]

for all \( f \leq 0 \) (4.1)

holds if and only if for all \( y \in \mathbb{R}^n \): \( \Phi(y)^p \leq \nu \), \( u = w^{1/q} \) and \( C = C(K, p, q) \) is a constant independent of \( f \).

**Proof.** The proof is immediate from Theorem 3.1. and Theorem 3.2, if we set \( K(x, y) \equiv \Phi(x - y) \).

**Corollary 4.2.** Suppose that \( w \geq 0 \) is locally integrable with respect to Lebesgue measure on \( \mathbb{R}_+ = (0, \infty) \). Denote the Laplace transform \( (L) \) of \( f \) on \( \mathbb{R}_+ \) by

\[
(Lf)(x) = \int_0^\infty e^{-xy} f(y) \, dy, \quad x \in \mathbb{R}_+
\]

for a fixed \( p \in (1, \infty) \). Then there exists \( \nu(x) < \infty \) almost everywhere and \( C > 0 \) such that

\[
\left[ \int_{\mathbb{R}^n} (Lf)^q w \, dx \right]^{1/q} \leq \left[ C \int_{\mathbb{R}^n} f^p \nu \, dx \right]^{1/p}
\]

for all \( f \leq 0 \)

holds if and only if \( (Lw)(x) < \infty, x \in \mathbb{R}_+ \).

**Comment.** There is a similar result for the dual operator as defined in [1].
References


