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THE CLAIRAUT AND LAGRANGE AREOLAR EQUATION

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Abstract. The method of differentiation and the Clairaut and Lagrange equations have not been considered for areolar equations, and thus the same is with the theory of singular integrals and singular points. A reason for this is that the areolar derivative $\dfrac{\partial W}{\partial \overline{z}}$ has not arithmetic properties of the usual quotient $\dfrac{df(z)}{dz}$ for analytic functions. In this paper we will try to solve equations with singular integrals for areolar equations and to begin qualitative and geometric theory.

1. PARTIAL AREOLAR DIFFERENTIAL

Areolar and generalized derivatives

$$\frac{\partial w}{\partial \overline{z}} \quad \text{and} \quad \frac{\partial w}{\partial z} \tag{1}$$

for complex functions of two independent variables do not have arithmetic properties of the usual quotient, which the derivative of an analytic function has:

$$\frac{df(z)}{dz} = f'(z),\ df = f'(z)dz,\ dz = \frac{df}{f'(z)},\ \frac{dz}{df} = \frac{1}{f'(z)}. \tag{2}$$

It is known that derivatives (1) are not only operational but also and partial, so that in the Riemann differential

$$dw = du + idv = \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \overline{z}}d\overline{z} \tag{3}$$

they play the same role as real partial derivatives, but the last ones have geometric property of direction coefficients of intersection curves, while for (1) such a simple interpretation is not possible. Already Riemann, observing that it is convinient to have some arithmetic properties of the usual quotient in the set $\{w(z, \overline{z})\}$ together with operational property of differentiation, introduced the *total derivative* dividing relation (3) by dz:

$$\frac{dw}{dz} = \frac{du + idv}{dx + idy} = \frac{\partial w}{\partial z} + \frac{d\overline{z}}{dz}\frac{\partial w}{\partial \overline{z}} = \frac{\partial w}{\partial z} + e^{-2id\varphi}\frac{\partial w}{\partial \overline{z}}. \tag{4}$$

This derivative has arithmetic properties as in (2), but it is not unique because it depends on $d\varphi$, that is on the path along which $z_0 + \Delta z \to z_0$.

Analogously with (4) we can define the *total areolar derivative* dividing (3) by $d\overline{z}$:

$$\frac{dw}{d\overline{z}} = \frac{du + idv}{dx - idy} = \frac{\partial w}{\partial \overline{z}} + \frac{dz}{d\overline{z}}\frac{\partial w}{\partial z} = \frac{\partial w}{\partial \overline{z}} + e^{2id\varphi}\frac{\partial w}{\partial z} \tag{5}$$

But, it is natural then to introduce also the total derivatives of conjugate values of the function:

$$\frac{d\overline{w}}{dz} = \frac{du - idv}{dx + idy} = \frac{\dfrac{\partial \overline{w}}{\partial z}dz + \dfrac{\partial \overline{w}}{\partial \overline{z}}d\overline{z}}{dz} = \frac{d\overline{w}}{dz} + e^{-2id\varphi}\frac{d\overline{w}}{d\overline{z}} \tag{6}$$

and

$$\frac{d\overline{w}}{d\overline{z}} = \frac{du - idv}{dx - idy} = \frac{\dfrac{\partial \overline{w}}{\partial z}dz + \dfrac{\partial \overline{w}}{\partial \overline{z}}d\overline{z}}{d\overline{z}} = \frac{d\overline{w}}{d\overline{z}} + e^{2id\varphi}\frac{\partial \overline{w}}{dz}. \tag{7}$$

For all different derivatives it holds:

$$\frac{\dfrac{dw}{dz}}{\dfrac{dw}{d\overline{z}}} : \frac{\dfrac{d\overline{w}}{dz}}{\dfrac{d\overline{w}}{d\overline{z}}} = \frac{\dfrac{du + idv}{dx + idy}}{\dfrac{du + idv}{dx - idy}} : \frac{\dfrac{du - idv}{dx + idy}}{\dfrac{du - idv}{dx - idy}} = 1,$$

which has some geometric interpretation that should be found.

We introduce the following notions:

Definition 1. *The partial areolar differential is defined as the usual product of areolar derivative* $\frac{\partial w}{\partial \overline{z}}$ *by* $d\overline{z}$*:*

$$d_{\overline{z}}w := \frac{\partial w}{\partial \overline{z}} d\overline{z} \tag{9}$$

and so

$$\frac{d_{\overline{z}}w}{d\overline{z}} = \frac{\partial w}{\partial \overline{z}}. \tag{10}$$

Definition 2. *The partial generalized differential is the product of generalized derivative by* dz*:*

$$d_z w := \frac{\partial w}{\partial z} dz \tag{11}$$

and so

$$\frac{d_z w}{dz} = \frac{\partial w}{\partial z}. \tag{12}$$

With these definitions the Riemann differential has the additivity property of the total differential:

$$dw = d_z w + d_{\overline{z}} w, \tag{13}$$

which is very important in technics. On the other hand, in this way we have possibilities for the reciprocal value and inversion, which implies the possibility to separate variables, and also we have similarity with the real differentials.

2. THE CLAIRAUT AREOLAR EQUATION

By analogy with the usual Clairaut differential equation, by which one introduces in the simplest way singular integral, we introduce the Clairaur areolar equation by

$$w = \overline{z}\frac{\partial w}{\partial \overline{z}} + f\left(\frac{\partial w}{\partial \overline{z}}\right), \tag{14}$$

where f is an analytic function of its argument.

If we differentiate (14) not totally, but partialy by $d\overline{z}$, we have

$$d_{\overline{z}}w = d_{\overline{z}}\left(\overline{z}\frac{\partial w}{\partial \overline{z}}\right) + d_{\overline{z}}\left(f\left(\frac{\partial w}{\partial \overline{z}}\right)\right), \tag{15}$$

from where in accordance to the definition (9):

$$\frac{\partial w}{\partial \overline{z}}d\overline{z} = \frac{\partial w}{\partial \overline{z}} + \overline{z}\frac{\partial^2 w}{\partial \overline{z}^2}d\overline{z} + f'\left(\frac{\partial w}{\partial \overline{z}}\right)\frac{\partial^2 w}{\partial \overline{z}^2}d\overline{z}$$

we obtain

$$\frac{\partial^2 w}{\partial \overline{z}^2}d\overline{z}\left(\overline{z} + f'\left(\frac{\partial w}{\partial \overline{z}}\right)\right) = 0.$$

Since $d\overline{z} \neq 0$, we only have

$$\frac{\partial^2 w}{\partial \overline{z}^2}\left(\overline{z} + f'\left(\frac{\partial w}{\partial \overline{z}}\right)\right) = 0. \tag{16}$$

From here we have only two possibilities:

1. $\dfrac{\partial^2 w}{\partial \overline{z}^2} = 0$: the equation of Goursat bianalytic functions:

$$\frac{\partial}{\partial \overline{z}}\left(\frac{\partial w}{\partial \overline{z}}\right) = 0, \quad \frac{\partial w}{\partial \overline{z}} = \varphi(z),$$

or

$$w = \overline{z}\varphi(z) + \psi(z). \tag{17}$$

However, the equation (14) is of the first order, and (19) can contain only one arbitrary function of two analytic functions $\varphi(z)$ and $\psi(z)$. If we substitute (17) in (14) we get

$$\overline{z}\varphi(z) + \psi(z) = \overline{z}(\varphi(z) + 0) + f(\varphi(z)),$$

from which we have

$$\psi(z) = f(\varphi(z)), \tag{18}$$

which is possible, since f, by assumption, is analytic, and the composition $f\varphi$ is also an analytic function. It follows:

Theorem 2.1. *The general solution of the Clairaut areolar equation* (14) *is given by*

$$w(z, \overline{z}) = \overline{z}\varphi(z) + f(\varphi(z)), \tag{19}$$

where $f(z)$ is an analytic function of its argument.

2. From (16) we have also

$$\overline{z} + f'\left(\frac{\partial w}{\partial \overline{z}}\right) = 0. \tag{20}$$

If we put $\dfrac{\partial w}{\partial \overline{z}} = p$, ($p$ is a complex parameter), from (20) and (14) we have

$$\overline{z} = -f'(p) \quad \text{and} \quad w = -pf'(p) + f(p),$$

i.e. we obtain one solution more in parametric form

$$\left.\begin{array}{rl} \overline{z} = & -f'(p), \\ w = & -pf'(p) + f(p). \end{array}\right\} \tag{21}$$

The solution (21) cannot be obtain from (19) (for some $\varphi(z)$), is independent from $\varphi(z)$, hence it is not a particular solution. Therefore:

Theorem 2.2. *The parametric solution* (21) *is the singular integral of the Clairaut equation* (14).

Example. *Solve the Clairaut equation*

$$w = \overline{z}\frac{\partial w}{\partial \overline{z}} + \left(\frac{\partial w}{\partial \overline{z}}\right)^2.$$

The general solution is

$$w(z, \overline{z}) = \overline{z}\varphi(z) + \varphi^2(z).$$

For the singular solution we have: Since $\dfrac{\partial w}{\partial \overline{z}} = p$, $f(p) = 2p$, from (20) we have $\overline{z} = -2p$; if we substitute this in the equation one obtains $w = -p^2$. If we eliminate p, by (21) we have

$$p = -\frac{\overline{z}}{2}, \quad w = -\frac{\overline{z}^2}{4}.$$

By direct verification one can see that $w = -\overline{z}^2/4$ is the singular integral indeed.

3. THE LAGRANGE AREOLAR EQUATION

The Lagrange areolar equation is defined (by analogy with the usual real equation) in the following way:

$$w = \overline{z} f\left(\frac{\partial w}{\partial \overline{z}}\right) + g\left(\frac{\partial w}{\partial \overline{z}}\right), \tag{22}$$

where f and g are analytic functions of $\dfrac{\partial w}{\partial \overline{z}}$.

Differentiating this equation partially by $\overline{z}$ we have

$$d_{\overline{z}} w = d_{\overline{z}}\left(\overline{z} f\left(\frac{\partial w}{\partial \overline{z}}\right) + g\left(\frac{\partial w}{\partial \overline{z}}\right)\right),$$

or

$$\frac{\partial w}{\partial \overline{z}} d\overline{z} = f\left(\frac{\partial w}{\partial \overline{z}}\right) d\overline{z} + \overline{z} f'\left(\frac{\partial w}{\partial \overline{z}}\right)\frac{\partial^2 w}{\partial \overline{z}^2} d\overline{z} + g'\left(\frac{\partial w}{\partial \overline{z}}\right)\frac{\partial^2 w}{\partial \overline{z}^2} d\overline{z}\,.$$

If we let $\dfrac{\partial w}{\partial \overline{z}} = p$, we shall have

$$p d\overline{z} = f(p) d\overline{z} + \left(\overline{z} f'(p)\frac{\partial p}{\partial \overline{z}} + g'(p)\frac{\partial p}{\partial \overline{z}}\right) d\overline{z}\,,$$

or

$$(p - f(p)) d\overline{z} = \overline{z} f'(p)\left(\frac{\partial p}{\partial \overline{z}} d\overline{z}\right) + g'(p)\left(\frac{\partial p}{\partial \overline{z}} d\overline{z}\right).$$

According to the definition of partial differential of p by $\overline{z}$ we have

$$d_{\overline{z}} p = \frac{\partial p}{\partial \overline{z}} dz$$

and from that

$$(p - f(p)) d\overline{z} = \overline{z} f'(p) d_{\overline{z}} p + g'(p) d_{\overline{z}} p,$$

or

$$(p - f(p)) d\overline{z} = (\overline{z} f'(p) + g'(p)) d_{\overline{z}} p, \tag{23}$$

where $d\overline{z}$ and $d_{\overline{z}} p$ are free differentials.

From (23) we have

$$\frac{d_{\overline{z}} p}{d\overline{z}} = \frac{p - f(p)}{\overline{z} f'(p) + g'(p)},$$

which is an areolar equation with respect to $p(z, \overline{z})$, but a composition because the unknown function p is the variable in some analytic functions $f(p)$ and $g(p)$. However, according to (9) and (10), (23) can be written in the form

$$\frac{d\overline{z}}{d_{\overline{z}}p} = \frac{\overline{z}f'(p) + g'(p)}{p - f(p)}, \tag{24}$$

and now we can regard $\overline{z}$ as a function of p and z. But, the form (23) says that the total derivative of $\overline{z}$ by p is a linear function of $\overline{z}$ with coefficients which are functions of $f(p)$ and $g(p)$, so that (24) is actually an areolar equation with respect to the inverse derivative $\dfrac{d\overline{z}}{dp}$:

$$\frac{d\overline{z}}{d_{\overline{z}}p} - \frac{f'(p)}{p - f(p)}\overline{z} - \frac{g'(p)}{p - f(p)} = 0.$$

We shall solve this equation using well known formula for linear equations:

$$\overline{z}(p, z) = e^{\int \frac{f'(p)}{p - f(p)} dp} \left[\varphi(z) + \int e^{-\int \frac{f'(p)}{p - f(p)} dp} \frac{g'(p)}{p - f(p)} dp\right], \tag{25}$$

where $\varphi(z)$ is an arbitrary analytic function of z. Then, by (22), we have

$$w = f(p)\overline{z}(p, z) + g(p). \tag{26}$$

It follows

Theorem 3.1. *The equations* (25) *and* (26) *give the parametric general solution of the Lagrange areolar equation* (22)*, where* $\varphi(z)$ *is an arbitrary analytic function of* z *in the role of a complex "integral constant".*

This procedure can be applied if

$$p - f(p) = 0, \tag{27}$$

that is

$$\frac{\partial w}{\partial \overline{z}} = f\left(\frac{\partial w}{\partial \overline{z}}\right).$$

Then we obtain

$$(\overline{z}f'(p) + g'(p))d_{\overline{z}}p = 0,$$

when p is in the role of independent variable, and $d_{\overline{z}}p \neq 0$. Further we have

$$\overline{z}f'(p) + g'(p) = 0,$$

from where

$$\overline{z} = -\frac{g'(p)}{f'(p)}, \tag{28}$$

and from (22),

$$w = -f(p)\frac{g'(p)}{f'(p)} + g(p). \tag{29}$$

The equations (28) and (29) are parametric solution of the equation (24), since the condition (27) is implied by (22) which is a special singular case. Moreover, it holds:

Theorem 3.2. *If the equations* (28) *and* (29) *give the solution of the equation* (22), *then they represent the singular solution of that equation.*

For $f(p) = p = \dfrac{\partial w}{\partial \overline{z}}$ the Lagrange equation (22) is the Clairaut equation (14).

Example. *Solve the areolar equation*

$$w = \overline{z}\left(\frac{\partial w}{\partial \overline{z}}\right)^2 + \frac{\partial w}{\partial \overline{z}} \tag{1}$$

Differentiating the equation by $\overline{z}$ we have

$$\frac{\partial w}{\partial \overline{z}}d\overline{z} = \left(\frac{\partial w}{\partial \overline{z}}\right)^2 d\overline{z} + 2\overline{z}\left(\frac{\partial w}{\partial \overline{z}}\right)\frac{\partial^2 w}{\partial \overline{z}^2}d\overline{z} + \frac{\partial^2 w}{\partial \overline{z}^2}d\overline{z}.$$

Letting $\dfrac{\partial w}{\partial \overline{z}} = p$ we have

$$pd\overline{z} = p^2 d\overline{z} + 2\overline{z}p\left(\frac{\partial p}{\partial \overline{z}}d\overline{z}\right) + \frac{\partial p}{\partial \overline{z}}d\overline{z},$$

i.e.

$$p(1-p)d\overline{z} = (2p\overline{z} + 1)d_{\overline{z}}p.$$

Because of arithmetic properties of the total derivative

$$\frac{d\overline{z}}{d_{\overline{z}}p} = \frac{2p\overline{z}}{p(1-p)} + \frac{1}{p(1-p)}$$

we get the linear (inverse) areolar equation

$$\frac{d\overline{z}}{d_{\overline{z}}p} - \frac{2}{1-p}\overline{z} - \frac{1}{p(1-p)} = 0,$$

whose solution is

$$\overline{z}(p,z) = e^{\int \frac{2}{1-p}dp}\left[\varphi(z) + \int e^{-\int \frac{2}{1-p}dp}\frac{dp}{p(1-p)}\right].$$

After short calculation we get

$$\overline{z}(p,z) = \frac{1}{(1-p)^2}\left[\varphi(z) + ln\ p - p\right].$$

So, the general solution of the equation (1) is

$$\left.\begin{aligned} \overline{z} = & \quad \frac{1}{(1-p)^2}\left[\varphi(z) + ln\ p - p\right], \\ w(z,\overline{z}) = & \quad \frac{p^2}{(1-p)^2}\left[\varphi(z) + ln\ p - p\right] + p. \end{aligned}\right\} \tag{2}$$

Verification. In order to show how the inverse of the total derivative can be use we make a verification.

From the first equality in $(**)$ we have

$$\overline{z}(1-p)^2 + 2\overline{z}(1-p)\left(-\frac{\partial p}{\partial \overline{z}}\right) = \left(\frac{1}{p} - 1\right)\frac{\partial p}{\partial \overline{z}},$$

and if we factorise this we have

$$(1-p)\left[(1-p) - 2\overline{z}\frac{\partial p}{\partial \overline{z}} - \frac{1}{p}\frac{\partial p}{\partial \overline{z}}\right] = 0,$$

from where

$$(1-p) - \left(2\overline{z} + \frac{1}{p}\right)\frac{\partial p}{\partial \overline{z}} = 0.$$

Differentiating now the second equation in (2)

$$w = p^2\overline{z} + p$$

by $\overline{z}$, one obtains

$$\frac{\partial w}{\partial \overline{z}} = p = p^2 + 2p\overline{z}\frac{\partial p}{\partial \overline{z}} + \frac{\partial p}{\partial \overline{z}},$$

or

$$p(1-p) = (2p\overline{z}+1)\frac{\partial p}{\partial \overline{z}}\,. \tag{3}$$

From here one has

$$\frac{\partial p}{\partial \overline{z}} = \frac{p(1-p)}{2p\overline{z}+1}\,.$$

If the last expression we substitute in (3) we shall see that is satisfied identically:

$$(1-p) - \frac{2p\overline{z}+1}{p}\frac{p(1-p)}{2p\overline{z}+1} \equiv 0,$$

so that the solution (2) satisfies the equation (1) for every $\varphi(z)$. This means that (2) is the general solution.

Remark. The consideration above does not hold for $p = 0$ and $p = 1$, i.e. for $\frac{\partial w}{\partial \overline{z}} = 0$ and $\frac{\partial w}{\partial \overline{z}} = 1$, when we obtain $w_1 = \varphi(z)$ and $w_2 = \overline{z} + \varphi(z)$.

If we substitute w_1 into the equation we have $\varphi(z) = 0$, i.e. $w_1 \equiv 0$. So, w_1 is the trivial solution.

If we substitute w_2 we have $\overline{z} + \varphi(z) = \overline{z} + 1$, i.e. we have $\varphi(z) = 1$.

The solution $w_2 = \overline{z} + 1$ has a special role; for this solution we have the following disscusion:

From (1) we have

$$\frac{\partial w}{\partial \overline{z}} = \frac{-1 \pm \sqrt{1+4\overline{z}w}}{2\overline{z}},$$

and for $w_2 = \overline{z} + 1$ one has $\frac{\partial w}{\partial \overline{z}} = 1$, which means that the previous equation is satisfied by this value only for the "+" case. Then that equation is

$$\frac{\partial w}{\partial \overline{z}} = \frac{2w}{\sqrt{1+4\overline{z}w}+1}.$$

We see that the point $\overline{z} = 0$ is only apparently a singular point of the equation. Besides, we conclude that if $4\overline{z}w + 1 = 0$ we have disscusion for ramification of the equation $w = -\frac{1}{4\overline{z}}$.

Open problems. It would be interesting to study the existence of singular points, points of ramification of solutions, geometrical sets of singular points, pseudoenvelopes. Geometry of areolar equations, in general, is not considered yet.

The Vekua-Clairaut and Vekua-Lagrange equations and their theory of singular points and singular integrals would be more interesting, and mathematically more important, for consideration.

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