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ABOUT POROSITY PARAMETERS WITH THE APPLICATION OF GENERAL SIMILARITY METHOD TO THE CASE OF A DISSOCIATED GAS FLOW IN THE BOUNDARY LAYER

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Abstract. In this paper we have obtained the momentum equation for the ideally dissociated gas (air) flow in the boundary layer in the case of a porous contour of the body within fluid. We have also defined the set of porosity parameters Λ_k that is necessary for the application of generalized similarity method in order to solve different problems of compressible fluid flow.

1. STARTING EQUATIONS

As it is known [2], [4], generalized similarity method is based on the use of the momentum equation and on the application of infinite parameters sets that are accepted for new independent variables. Depending on the studied flow problem (incompressible fluid, MHD boundary layer, dissociated gas, ...), these parameters have different forms and they are related with different recurrent simple differential equations [2]. In this paper we have derived this momentum equation with the ideally dissociated gas (air) flow in the boundary layer in the case when the contour of the body within fluid is **porous**. At the same time, a porosity parameter has been introduced and a set of porosity parameters of the ideally dissociated gas has been defined. A complete system of equations of the laminar planar and steady boundary layer of the ideally dissociated gas (air) [3] is:

$$
\begin{aligned}
&\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho \nu)=0,\\
&\rho u\frac{\partial u}{\partial x}+\rho\nu\frac{\partial u}{\partial y}=\rho_e u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),\\
&\rho u\frac{\partial \alpha}{\partial x}+\rho\nu\frac{\partial \alpha}{\partial y}=\frac{\partial}{\partial y}\left(\rho D\frac{\partial \alpha}{\partial y}\right)+\dot{W}_A,\\
&\rho c_p\left(u\frac{\partial T}{\partial x}+\nu\frac{\partial T}{\partial y}\right)=-u\rho_e u_e\frac{du_e}{dx}+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)\\
&-(h_A-h_M)\dot{W}_A+\rho D(c_{pA}-c_{pM})\frac{\partial \alpha}{\partial y}\frac{\partial T}{\partial y},\\
&p=\rho T(1+\alpha)\frac{k}{2m_A}=\rho T(1+\alpha)R_M;
\end{aligned}
\tag{1}
$$

These equations represent respectively: the continuity equation, the dynamic equation, the equation of diffusion of the atomic component of the ideally dissociated gas, the energy equation and the state equation. The notation usual in the boundary layer theory has been used for the physical values in the system of equations (1): $u(x,y)$ – longitudinal projection of the velocity in the boundary layer, $\nu(x,y)$ – transversal projection of the velocity, ρ – density, p – pressure, μ – dynamic viscosity coefficient, α – mass concentration of the ideally dissociated gas atomic component, D – coefficient of the diffusion, $\dot{W}_A$ – mass formation rate due to dissociation of gas molecules, c_p – specific heat of the gas (as a mixture), k – Boltzmann constant, λ – coefficient of thermal conductivity, T – absolute temperature, m – atomic and molecular mass of the ideally dissociated gas, R – gas constant and h – enthalpy. The subscripts represent: e – conditions at the outer edge of the boundary layer and A, M – atomic and molecular component of the ideally dissociated gas, respectively.

If, with ν_w, we denote the given velocity of the dissociated gas flow trough the solid porous wall of the body within fluid ($\nu_w > 0$ at injection, $\nu_w < 0$ at ejection) and transversal to it, then the corresponding boundary conditions are:

$$\begin{aligned} &u = 0, \quad \underline{\nu = \nu_w(x)}, \quad T = T_w, \quad \alpha = \alpha_w \quad \text{for} \quad y = 0, \\ &u \to u_e(x), \qquad T \to T_e(x), \quad \alpha \to \alpha_e(x) \quad \text{for} \quad y \to \infty. \end{aligned} \tag{2}$$

Here and further on, the subscript w stands for the physical values at the wall of the body within fluid.

2. MOMENTUM EQUATION OF THE CONSIDERED PROBLEM

In order to obtain the momentum equation of the dissociated gas in the case of a porous contour of the body within fluid ($\nu_w(x) \neq 0$), we start only from the continuity equation and the dynamic equation of the boundary layer of the system (1) of the considered problem.

If the continuity equation is multiplied with $u_e(x)$, where $u_e(x)$ is the known velocity of the ideally dissociated gas at the outer edge of the boundary layer, then this equation comes down to:

$$\frac{\partial}{\partial x}(\rho u u_e) + \frac{\partial}{\partial y}(\rho \nu u_e) = \rho u \frac{du_e}{dx} \, .$$

Because of the continuity equation, the dynamic equation of the system (1) can be transformed into:

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho u \nu) = \rho_e u_e \frac{du_e}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \, .$$

Subtracting these equations we get:

$$\frac{\partial}{\partial x}(\rho u u_e - \rho u^2) + \frac{\partial}{\partial y}(\rho \nu u_e - \rho u \nu) = \frac{du_e}{dx}(\rho u - \rho_e u_e) - \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \, .$$

Analogous to the way we derive the momentum equation of incompressible fluid, the previous equation can be integrated term by term with respect to the variable y, starting from the inner to the outer edge of the boundary layer, i.e., transversally to the layer. Taking the boundary conditions into consideration (2) and presuming the possibility of changing the order of differentiation and integration, we obtain:

$$\frac{d}{dx}\left[\int_0^\infty \rho u(u_e - u)\,dy\right] - \underline{\rho_w \nu_w u_e} = \frac{du_e}{dx}\int_0^\infty (\rho u - \rho_e u_e)\,dy + \left(\mu \frac{\partial u}{\partial y}\right)_{y=0} . \tag{3}$$

This equation represents Kármán's integral relation for the considered case of the ideally dissociated gas flow.

In order for this momentum equation of the considered problem to be of the same form as the momentum equation of the corresponding problem of incompressible fluid flow, it is necessary to introduce new variables. So, analogous to other compressible fluid flow problems [3], instead of the physical coordinates x and y, we introduce a new longitudinal $s(x)$ and a new transversal variable $z(x, y)$ in the form of the following relations:

$$s(x) = \frac{1}{\rho_0 \mu_0}\int_0^x \rho_w \mu_w\,dx, \qquad z(x, y) = \int_0^y \frac{\rho}{\rho_0}\,dy. \tag{4}$$

In the transformations (4), ρ_0 and μ_0 represent an approximate known value of the density and the coefficient of dynamic viscosity, while $\rho_w(x)$ and $\mu_w(x)$ are their the given known values at the inner edge of the boundary layer.

By means of the newly introduced variables, the equation (3) is brought to:

$$\frac{d}{ds}\left(u_e^2 \Delta^{**}\right) + u_e \frac{du_e}{ds}\Delta^* = \underline{\frac{\rho_w \nu_w u_e}{\rho_0 ds/dx}} + \frac{\tau_w}{\rho_0 ds/dx}\,, \tag{5}$$

where the conditional displacement thickness Δ^*, the conditional momentum loss thickness Δ^{**}, tangential stress at the wall of the body within fluid τ_w, and non-dimensional function of the friction ζ are determined as:

$$\Delta^*(s) = \int_0^\infty \left(\frac{\rho_e}{\rho} - \frac{u}{u_e}\right) dz, \quad \Delta^{**}(s) = \int_0^\infty \frac{u}{u_e}\left(1 - \frac{u}{u_e}\right) dz,$$

$$\tau_w = \left(\mu \frac{\partial u}{\partial y}\right)_{y=0} = \frac{\rho_w \mu_w}{\rho_0}\frac{u_e}{\Delta^{**}}\zeta, \quad \zeta(s) = \left[\frac{\partial(u/u_e)}{\partial(z/\Delta^{**})}\right]_{z=0}. \tag{6}$$

After a little simpler transformations, the equation (5) can be brought to:

$$\frac{d}{ds}\left(\frac{{\Delta^{**}}^2}{\nu_0}\right) + \frac{2}{u_e}\frac{u_e' {\Delta^{**}}^2}{\nu_0}\left(2 + \frac{\Delta^*}{\Delta^{**}}\right) = \frac{2}{u_e ds/dx}\left(\frac{\rho_w \mu_w}{\rho_0 \mu_0}\zeta + \frac{\rho_w \nu_w \Delta^{**}}{\rho_0 \nu_0}\right), \tag{7}$$

where " $'$ " stands for the derivation with respect to the longitudinal variable s, while $\nu_0 = \mu_0/\rho_0$.

Introducing the common symbols and the parameter of the form f:

$$Z^{**} = \frac{{\Delta^{**}}^2}{\nu_0}, \quad f = u_e' {\Delta^{**}}^2 \nu_0 = u_e' Z^{**} = f_1; \quad \frac{\Delta}{\Delta^{**}} = H, \tag{8}$$

the equation (7) reduces to it is final form

$$\frac{dZ^{**}}{ds} = \frac{F_{dp}}{u_e}. \tag{9}$$

It represents the momentum equation of the dissociated gas and it has the same form as the momentum equation of incompressible fluid. The characteristic function F_{dp} of the dissociated gas in the case of a porous contour, in the equation (9), is determined with the expression:

$$F_{dp} = 2[\zeta - (2 + H)f] + 2\underline{\frac{\mu_0}{\mu_w}\frac{\nu_w \Delta^{**}}{\nu_0}}. \tag{10}$$

It is noticed that the function F_{dp}, compared to the corresponding function in the case of a non-porous contour [3], contains an additional term. The given velocity ν_w figures in this term, and it depends on the porosity of the contour of the body within fluid. Because of this a new parameter has been introduced in this paper – **the porosity parameter**. It is characterized by injection (or ejection) of the dissociated gas into the boundary layer, and it is determined with the expression:

$$\Lambda(s) = -\frac{\mu_0}{\mu_w}\frac{\nu_w \Delta^{**}}{\nu_0} = -\frac{V_w \Delta^{**}}{\nu_0} = \Lambda_1, \quad V_w = \frac{\mu_0}{\mu_w}\nu_w\,, \tag{11}$$

where V_w is the conditional velocity of injection.

In this way, the function F_{dp} comes down to:

$$F_{dp} = 2[\zeta - (2+H)f] - 2\Lambda. \tag{12}$$

In this form it will be used in our further studies of the dissociated gas boundary layer.

3. INTRODUCTION OF A SET OF POROSITY PARAMETERS

The obtained momentum equation (9) can be written in jet another form. If the value Z^{**} is expressed by means of the parameter of the form (8) $Z^{**} = f/u_e'$, and if the differentiation with respect to s is performed, the momentum equation will be obtained in the following form:

$$\frac{df}{ds} = \frac{u_e'}{u_e}F_{dp} + \frac{u_e''}{u_e'}f \tag{13}$$

which is more convenient for application of the general similarity method.

Because of the relation (8) between the value Z^{**} and the conditional thickness Δ^{**}, the porosity parameter (11) can be also written as:

$$\Lambda = -\frac{V_w}{\sqrt{\nu_0}}{Z^{**}}^{1/2} = \Lambda_1 \tag{14}$$

where it follows that:

$$Z^{**} = \frac{\Lambda^2 \nu_0}{V_w^2}\,.$$

Based on this relation, from the momentum equation (9) we obtain:

$$\frac{d\Lambda}{ds} = \frac{d\Lambda_1}{ds} = \frac{u_e'}{u_e f_1}\left(\frac{1}{2}F_{dp}\Lambda_1 - u_e\frac{V_w'}{\sqrt{\nu_0}}{Z^{**}}^{3/2}\right)\,.$$

If the second addend in brackets is declared to be a new parameter Λ_2

$$\Lambda_2 = -u_e\frac{V_w'}{\sqrt{\nu_0}}{Z^{**}}^{3/2}\,, \tag{15}$$

the previous equation takes this form:

$$\frac{u_e}{u_e'} f_1 \frac{d\Lambda_1}{ds} = \frac{1}{2} F_{dp} \Lambda_1 + \Lambda_2 \equiv \chi_1 \,. \tag{16}$$

Differentiating the parameter Λ_2 with respect to the variable s, we will obtain:

$$\frac{u_e}{u_e'} f_1 \frac{d\Lambda_2}{ds} = \left(f_1 + \frac{3}{2} F_{dp} \right) \Lambda_2 + \Lambda_3 \equiv \chi_2 \,, \tag{17}$$

where the value

$$\Lambda_3 = -u_e^2 \frac{V_w''}{\sqrt{\nu_0}} Z^{**^{5/2}} \,, \tag{18}$$

is taken as the third porosity parameter.

If the parameter Λ_3 is also differentiated with respect to the longitudinal variable s, we will get the equation:

$$\frac{u_e}{u_e'} f_1 \frac{d\Lambda_3}{ds} = \left(2f_1 + \frac{5}{2} F_{dp} \right) \Lambda_3 + \Lambda_4 \equiv \chi_3 \,, \tag{19}$$

where

$$\Lambda_4 = -u_e^3 \frac{V_w'''}{\sqrt{\nu_0}} Z^{**^{7/2}} \,. \tag{20}$$

Continuing the shown procedure, with the next parameters and based on (14), (15), (18) and (20), it is easily concluded that the general porosity parameter $\Lambda_k(s)$ is determined with the next expression

$$\Lambda_k = -u_e^{k-1} \left(\frac{V_w}{\sqrt{\nu_0}} \right)^{(k-1)} Z^{**^{k-1/2}}, \quad (k = 1, 2, 3, \ldots). \tag{21}$$

As the analysis indicates, any porosity parameter satisfies the recurrent equation

$$\frac{u_e}{u_e'} f_1 \frac{d\Lambda_k}{ds} = \{(k-1)f_1 + [(2k-1)/2] F_{dp}\} \, \Lambda_k + \Lambda_{k+1} \equiv \chi_k \,. \tag{22}$$

whose right hand-side is marked with χ_k $(k = 1, 2, 3, \ldots)$ in order to make it shorter.

Finally, we must point out the necessity to introduce the set of porosity parameters (21) in order to apply the general similarity method by means of which the starting system of equations (1) of the ideally dissociated gas flow in the case of a porous contour of the body within fluid, is brought to a general form. This set of parameters of Loitsianskii's type, plays the role of new independent variables.

It is particulary important that the set of parameter Λ_k, obtained (defined) in this paper has the identical form as the corresponding set of parameters λ_k in the case of incompressible fluid flow [1], [2]. The corresponding expressions for the characteristic function F_{dp} and χ_k are also identical. Of course, here the porosity parameters are expressed by means of the newly introduced variable s instead of the physical variable x, which is the case with incompressible fluid flow. Further more, in the conditions of incompressible fluid flow ($\rho, \mu =$ const.), the newly introduced variables (4) come down to $s(x) \to x$; $z(x, y) \to y$, where the porosity parameter comes down to $\Lambda_1 \to \lambda_1$ because in that case the conditional velocity of injection is $V_w \to \nu_w$.

These facts prove the correctness of defining sets of porosity parameters of the ideally dissociated gas in the form of the expressions (21). That is why in our further studies of dissociated gas flow in the case of a porous contour of the body within fluid, we shall use a set of porosity parameters in the form of the expression (21).

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