Kragujevac J. Math. 30 (2007) 343–354.

APPLICATION OF DIFFERENTIAL EQUATIONS

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(Received January 19, 2007)

Abstract. This paper considers dynamic reactions on wheels during braking of the vehicle. The braking vehicle is modelled by the system of differential equations. The solutions of the system are numerically determined. The theoretical method is verified by experimental results. The parameters relevant for the vehicle stability at braking are obtained by means of the results obtained.

1. INTRODUCTION

In dynamics of motor vehicles it is usual that dominant vehicle motions are mutual separated and that the straight motion of a vehicle with certain problems are observed.

In practice motions of vehicles caused by various causes of coupling, therefore making of complex models are done. In this paper the coupling is considered between the vertical motion being caused by uneven roads and the vertical motion caused by the vehicle braking.

In the paper the following designations are used:

$x[m]$ - vehicle longitudinal coordinate

$y[m]$ - vehicle cross coordinate

$z[m]$ - vehicle vertical coordinate

$\theta[rad]$ - rotation angle of vehicle center around cross axle

$a[m]$ - distance of vehicle center from front axle

$b[m]$ - distance of vehicle center from rear axle

$h[m]$ - height of vehicle center

$g[m/s^2]$ - gravitation acceleration

$z_1[m]$ - displacement of abut mass on front axle

$z_2[m]$ - displacement of abut mass on rear axle

$z_{1p}[m]$ - displacement of front wheel center

$z_{2p}[m]$ - displacement of rear wheel center

$z_{01}[m]$ - uneven road in point of contact of front wheel and road

$z_{02}[m]$ - uneven road in point of contact of rear wheel and road

$m[kg]$ - vehicle abut mass

$m_1[kg]$ - abut mass on front axle

$m_2[kg]$ - abut mass on rear axle

$m_{1p}[kg]$ - non abut mass on front axle

$m_{2p}[kg]$ - non abut mass on rear axle

$I_y[kgm^2]$ - inertia moment for center y-line

$K_1[Ns/m]$ - characteristics of front suspension damping

$K_2[Ns/m]$ - characteristics of rear suspension damping

$C_1[N/m]$ - characteristics of front suspension rigidity

$C_2[N/m]$ - characteristics of rear suspension rigidity

$C_{1p}[N/m]$ - characteristics of front wheel rigidity

$C_2[N/m]$ - characteristics of rear wheel rigidity

$F_x[N]$ - total longitudinal force

$F_{x_1}[N]$ - brake force on front wheels

$F_{x_2}[N]$ - brake force on rear wheels

$F_z[N]$ - total vertical force

$F_{z_1}[N]$ - vertical force on front axle

$F_{z_2}[N]$ - vertical force on rear axle

$\frac{\partial x}{\partial z}|_1[\cdots]$ - derivative of front suspension, gradient of change of longitudinal displacement related to vertical one

$\frac{\partial x}{\partial z}|_2[\cdots]$ - derivative of rear suspension, gradient of change of longitudinal displacement related to vertical one

$\varphi[\cdots]$ - coefficient of vehicle longitudinal adherence

Vehicle braking by ideal curve of vehicle braking on figure 1.

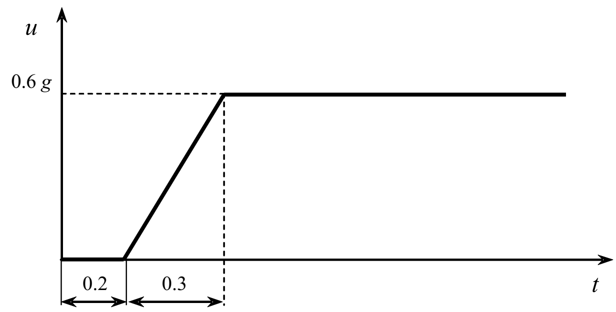

Figure 1: Diagram of vehicle slowing down

Brake force can be described by the equation:

$$F_x = \begin{cases} F_x = 0, & \text{if } 0\text{s} \leq t \leq 0.2\text{s}, \\ F_x = \frac{0.6gm}{0.3}(t - 0.3), & \text{if } 0.2\text{s} \leq t \leq 0.5\text{s}, \\ F_x = 0.6gm, & \text{if } t > 0.5\text{s}, \end{cases} \tag{1}$$

Uneven road in contact points of front and rear wheel is described by random function (RND) amplitude $10mm$:

$$\begin{aligned} z_{01} &= 0.01(RND - 0.5) \\ z_{02} &= 0.01(RND - 0.5) \end{aligned} \tag{2}$$

2. MATHEMATICAL MODEL OF BRAKED VEHICLE

For analyze of braked vehicle stability the braked vehicle model was observed as presented on figure 2 where it is seen that effect of rigidity of front and rear wheel and uneven road are monitored.

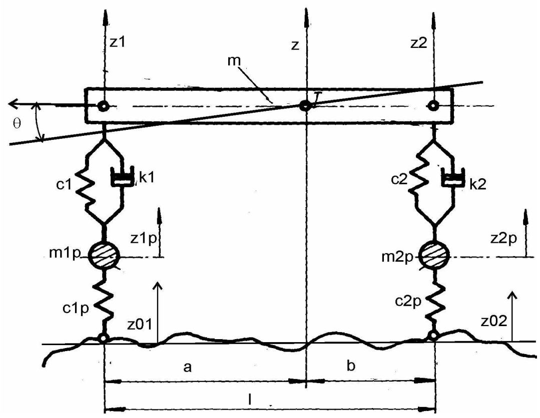

Figure 2: Mathematical model of braking vehicle

Appropriate system of differential equations are as follows:

$$\begin{aligned}
&m\ddot{x} = F_x \\
&m\ddot{z} + k_1(\dot{z}_1 - \dot{z}_{1p}) + c_1(z_1 - z_{1p}) + k_2(\dot{z}_2 - \dot{z}_{2p}) + c_2(z_2 - z_{2p}) = F_z \\
&I_y\ddot{\theta} + a[k_1(\dot{z}_1 - \dot{z}_{1p}) + c_1(z_1 - z_{1p})] - b[k_2(\dot{z}_2 - \dot{z}_{2p}) + c_2(z_2 - z_{2p})] = F_\theta \\
&m_{1p}\ddot{z}_{1p} - k_1(\dot{z}_1 - \dot{z}_{1p}) - c_1(z_1 - z_{1p}) + c_{1p}(z_{1p} - z_{01}) = 0 \\
&m_{2p}\ddot{z}_{2p} - k_2(\dot{z}_2 - \dot{z}_{2p}) - c_2(z_2 - z_{2p}) + c_{2p}(z_{2p} - z_{02}) = 0
\end{aligned} \tag{3}$$

Since

$$\begin{aligned}
z_1 &= z - a\theta \qquad & \dot{z}_1 &= \dot{z} - a\dot{\theta} \\
z_2 &= z + b\theta \qquad & \dot{z}_2 &= \dot{z} + b\dot{\theta}
\end{aligned} \tag{4}$$

the system 3 has the following form:

$$
\begin{aligned}
& m\ddot{x} = F_x \\
& m\ddot{z} + (k_1 + k_2)\dot{z} + (-k_1 a + k_2 b)\dot{\theta} - k_1 \dot{z}_{1p} - k_2 \dot{z}_{2p} \\
& \qquad\qquad + (c_1 + c_2)z + (-c_1 a + c_2 b) - c_1 z_{1p} - c_2 z_{2p} = F_z \\
& I_y \ddot{\theta} + (ak_1 - bk_2)\dot{z} + (-a^2 k_1 - b^2 k_2)\dot{\theta} - ak_1 \dot{z}_{1p} + bk_2 \dot{z}_{2p} \\
& \qquad\qquad + (ac_1 - bc_2)z + (-a^2 c_1 - b^2 c_2)\theta - ac_1 z_{1p} + bc_2 z_{2p} = F_\theta \\
& m_{1p}\ddot{z}_{1p} - k_1 \dot{z} + k_1 a\dot{\theta} + k_1 \dot{z}_{1p} - c_1 z + c_1 a\theta + (c_1 + c_{1p})z_{1p} = c_{1p} z_{01} \\
& m_{2p}\ddot{z}_{2p} - k_2 \dot{z} - k_2 b\dot{\theta} + k_2 \dot{z}_{2p} - c_2 z - c_2 b\theta + (c_2 + c_{2p})z_{2p} = c_{2p} z_{02}
\end{aligned} \tag{5}
$$

In matrix form is as follows:

$$
\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & I_y & 0 & 0 \\ 0 & 0 & 0 & m_{p1} & 0 \\ 0 & 0 & 0 & 0 & m_{2p} \end{bmatrix}
\begin{bmatrix} \ddot{x} \\ \ddot{z} \\ \ddot{\theta} \\ \ddot{z}_{1p} \\ \ddot{z}_{2p} \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & k_1 + k_2 & -k_1 a + k_2 b & -k_1 & -k_2 \\ 0 & ak_1 - bk_2 & -a^2 k_1 - b^2 k_2 & -ak_1 & bk_2 \\ 0 & -k_1 & ak_1 & k_1 & 0 \\ 0 & -k_2 & -bk_2 & 0 & k_2 \end{bmatrix}
\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{\theta} \\ \dot{z}_{1p} \\ \dot{z}_{2p} \end{bmatrix}
+
$$
$$
+
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 + c_2 & -c_1 a + c_2 b & -c_1 & -c_2 \\ 0 & ac_1 - bc_2 & -a^2 c_1 - b^2 c_2 & -ac_1 & -bc_2 \\ 0 & -c_1 & -ac_1 & c_1 + c_{1p} & 0 \\ 0 & -c_2 & -bc_2 & 0 & c_2 + c_{2p} \end{bmatrix}
\begin{bmatrix} x \\ z \\ \theta \\ z_{1p} \\ z_{2p} \end{bmatrix}
=
\begin{bmatrix} F_x \\ F_z \\ F_\theta \\ F_{1p} \\ F_{2p} \end{bmatrix} \tag{6}
$$

where:

$$
\begin{bmatrix} F_x \\ F_z \\ F_\theta \\ F_{1p} \\ F_{2p} \end{bmatrix}
=
\begin{bmatrix} F_x \\ F_{x_1} \frac{\partial x}{\partial z}|_1 + F_{x_2} \frac{\partial x}{\partial z}|_2 \\ -aF_{x_1} \frac{\partial x}{\partial z}|_1 + bF_{x_2} \frac{\partial x}{\partial z}|_2 + (F_{x_1} + F_{x_2})h \\ c_{1p} z_{01} \\ c_{2p} z_{02} \end{bmatrix} \tag{7}
$$

3. CALCULATION RESULTS

Previous differential equation system, with inputs from vehicle F and vehicle S, is solved by the numerical method Runge - Kutta. The next results are obtained.

Vehicle S is with transversal front and transversal rear wheel guiding:

$$
\frac{\partial x}{\partial z}|_1 = 0 \qquad \frac{\partial x}{\partial z}|_2 = 0 \tag{8}
$$

Vehicle F is with transversal front and longitudinal rear wheel guiding:

$$\frac{\partial x}{\partial z}|_1 = 0 \qquad \frac{\partial x}{\partial z}|_2 = -0.3 \tag{9}$$

Figures 3 and 4 show the road roughness at contacts between road and front and rear tyre, (z_{01} and z_{02}), respectively. These functions are given by stochastic random function.

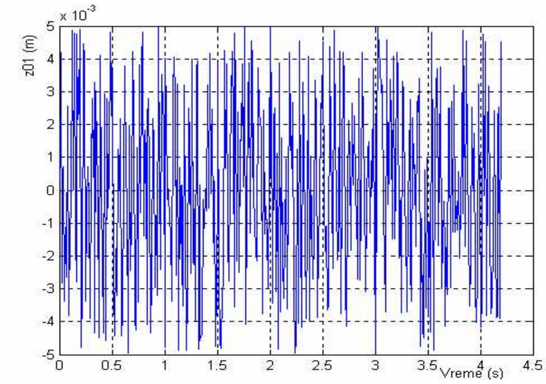

Figure 3: Function of uneven road in the points of contacts of front wheel and the ground

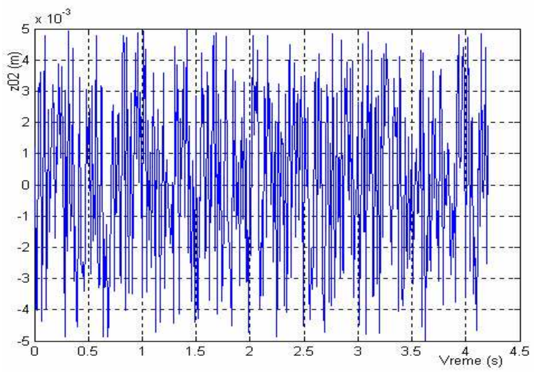

Figure 4: Function of uneven road in the points of contacts of rear wheel and the ground

3.1. CALCULATION RESULTS FOR THE VEHICLE F

The calculation results for the vehicle F are diagrams shown on figures from 5 to 8.

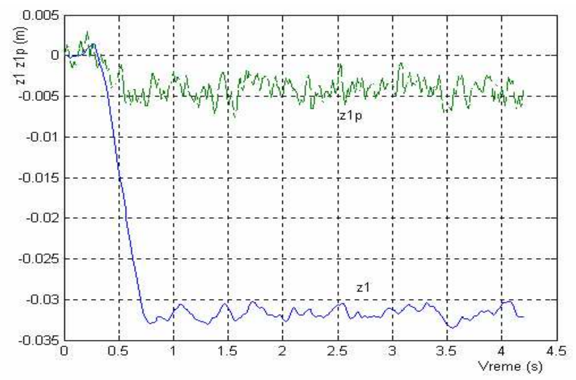

Figure 5: Displacement of abut mass and front wheel center of the vehicle F

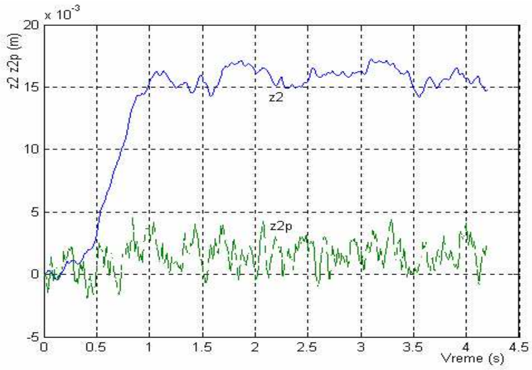

Figure 6: Displacement of abut mass and rear wheel center of the vehicle F

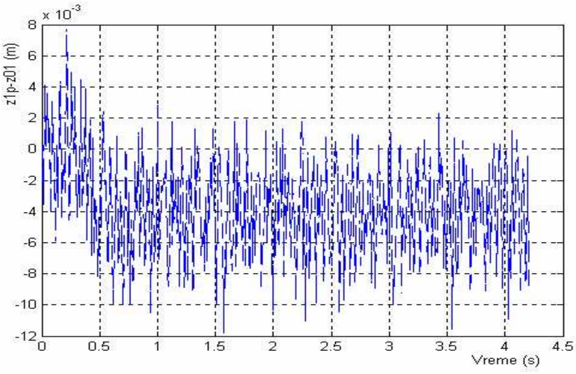

Figure 7: Difference of displacement of wheel center and uneven road on front axle of the vehicle F

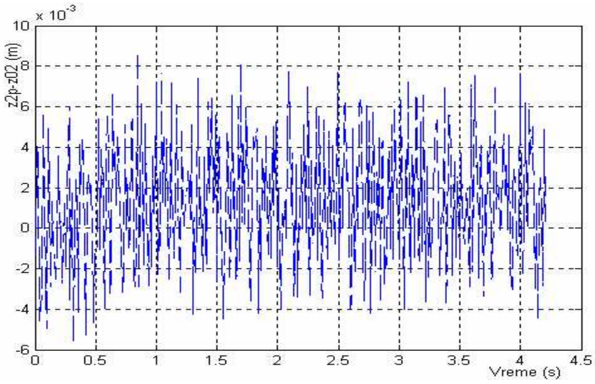

Figure 8: Difference of displacement of wheel center and uneven road on rear axle of the vehicle F

3.2. CALCULATION RESULTS FOR THE VEHICLE S

The calculation results for the vehicle S are diagrams shown on figures from 9 to 12.

Functions of uneven road in the points of contacts of front and rear wheels and the ground (z_{01} and z_{02}) are given by random functions (RND) being shown on figures 3 and 4 . Parallel review of displacement of the front wheel center and abut mass on front axle and the rear wheel center and abut mass on rear axle for the vehicles F and S are shown on the figures 5, 6, 9 and 10. The difference of displacement of

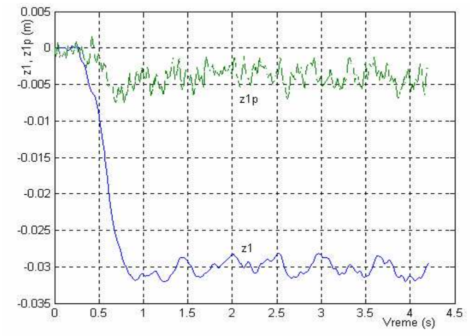

Figure 9: Displacement of abut mass and front wheel center of the vehicle S

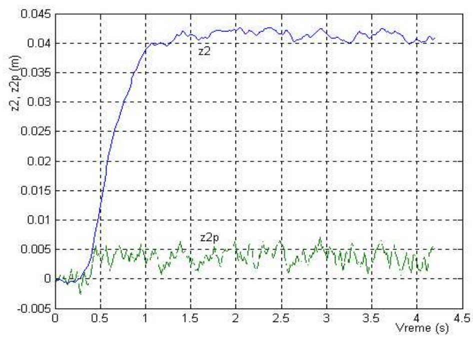

Figure 10: Displacement of abut mass and rear wheel center of the vehicle S

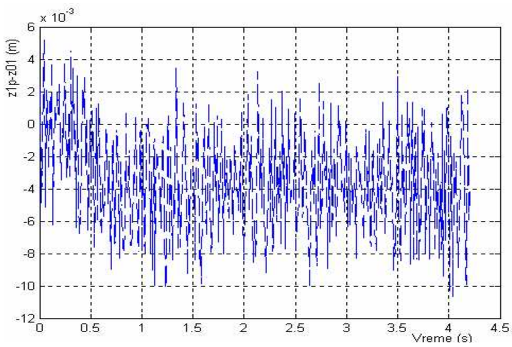

Figure 11: Difference of displacement of wheel center and uneven road on front axle of the vehicle S

front wheel center and uneven road and the rear wheel center and uneven road for the vehicles F and S are shown on figures 7, 8, 11 and 12.

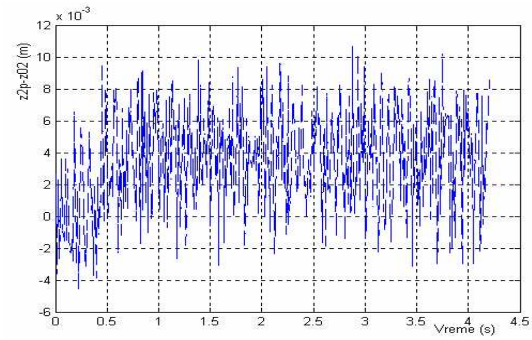

Figure 12: Difference of displacement of wheel center and uneven road on rear axle of the vehicle S

4. EFFECTIVE ANALYZE OF CALCULATED RESULTS

Since the time curve of displacement is in the discrete form the effective value is therefore calculated by the following relation:

$$z_{eff} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} z_i^2(t)} \tag{10}$$

where N is number of samples (number of points of signal discretion) and $z(t)$ is value of i-th sample (of i discrete value).

Lets define the effective value of the difference of displacement on front wheel:

$$(z_{1p} - z_{01})_{eff} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(z_{1p_i} - z_{01_i})^2} \tag{11}$$

and the effective value of displacement on rear wheel:

$$(z_{2p} - z_{02})_{eff} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(z_{2p_i} - z_{02_i})^2} \tag{12}$$

From diagrams on figures 7, 8, 11 and 12 we calculate effective values of displacements in time domain. The reached results are as follows:

$$\begin{array}{lll} \text{For the vehicle F:} & (z_{1p} - z_{01})_{eff} = 3.85mm & (z_{2p} - z_{02})_{eff} = 1.76mm \\ \text{For the vehicle S:} & (z_{1p} - z_{01})_{eff} = 3.72mm & (z_{2p} - z_{02})_{eff} = 4.05mm \end{array} \tag{13}$$

Based on calculated values it is seen the approximate equality of effective values on the front wheel and the difference of effective values on the rear wheel. Since in both cases the pneumatic is the same (the same rigidity $c_{2p} = 170N/mm$), thus higher effective value of displacement on the vehicle S gives higher dynamics forces. For example on the rear wheel of the vehicle S:

$$F_{d_2} = c_{2p}(z_{2p} - z_{02}) = 170 \cdot 4.05 = 688.5N \tag{14}$$

This force balances the rear axle and considerably reduces stability of the braked vehicle. On the vehicle F the relevant displacement is $(z_{2p} - z_{02})_{eff} = 1.76mm$, therefore the dynamics force is lower:

$$F_{d_2} = c_{2p}(z_{2p} - z_{02}) = 170 \cdot 1.76 = 299.2N \tag{15}$$

Therefore the vehicle F is considerably more stable at braking.

5. CONCLUSION

Fundamental theoretical approach which consider all relevant quantities for braking vehicle, as a result, gives effects of concepts for front and rear suspension system on the longitudinal vehicle stability.

The calculations with input data for vehicle F and vehicle S are done, and it can be seen that there are differences in displacements of rear axle with respect to the car body because of different wheel guiding.

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