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CONNECTIONS BETWEEN CUTS AND MAXIMUM SEGMENTS

Mirjana Borisavljević

Faculty of Transport and Traffic Engineering, University of Belgrade, Vojvode Stepe 305, 11000 Belgrade, Serbia
(email: mirjanab@afrodita.rcub.bg.ac.rs)

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Abstract. Pairs of systems, which consist of a system of sequents and a natural deduction system for some parts of intuitionistic logic, will be considered. For some of these pairs of systems the well-known property that cut-free sequent derivations correspond to normal derivations in natural deduction will be improved. In derivations of the systems of sequents a special kind of cuts, maximum cuts, will be defined. It will be shown that sequent derivations with cuts which are not maximum cuts correspond to normal derivations in natural deduction.

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1. INTRODUCTION

In [6] Gentzen introduced the system of sequents and the natural deduction system for intuitionistic logic (the systems LJ and NJ) and classical logic (the systems LK and NK). In the systems of sequents cut-free derivations, i.e. derivations without cuts, are the most important derivations. The most important natural deduction derivations are normal derivations, i.e. derivations without maximum segments. The main theorem of the systems of sequents is the cut-elimination theorem, and in the natural deduction systems the normalization theorem is the central one.

The problems of the connection between derivations of the systems of sequents and natural deduction systems, and the problems of the connection between reductions which constitute cut-elimination and normalization procedures are well known (see the Introductions of [4] and [14]). To solve these problems different pairs of systems of sequents and natural deduction systems, which are modifications of Gentzen's systems, were defined in [1], [8], [9], [10] and [14]. In almost all of these papers the connections between cut-free derivations and normal derivations were studied. From the results of these papers it can seem that "cut-free derivations and normal derivations are the same". However, the connection is the following:

the image of a cut-free derivation is a normal derivation,

(*) but

if a normal derivation is the image of a sequent derivation,

then that sequent derivation can have some cuts which can be eliminated.

From this non-symmetrical picture we can conclude that sequent derivations which correspond to normal derivations can have some cuts. In [2] and [4] Zucker's systems for intuitionistic predicate logic from [14], the system of sequents $\mathcal{S}$ and the natural deduction system $\mathcal{N}$, were considered. In derivations of the system $\mathcal{S}$ a special kind of cuts, maximum cuts, was defined. Roughly speaking, maximum cuts are cuts whose left cut formula is connected with the principal formula of a right rule (i.e. an introduction rule of a connective or a quantifier) and its right cut formula is connected with the principal formula of a left rule (i.e. an elimination rule of the connective or the quantifier). It was shown the following:

the image of a derivation without maximum cuts is a normal derivation,

(**) and

if a normal derivation is the image of a sequent derivation,

then that sequent derivation does not have maximum cuts.

In [5] the systems which introduced in [1], the system of sequents $\mathcal{SE}$ and the natural deduction system $\mathcal{NE}$, were considered. The system $\mathcal{SE}$ is very similar to Zucker's system $\mathcal{S}$ from [14]. Namely, Zucker's system $\mathcal{S}$ is the system $\delta\mathcal{E}$ from [1] whose formulae have only lower indices and the system $\mathcal{SE}$ is that system $\delta\mathcal{E}$ whose formulae have only upper indices. The system $\mathcal{NE}$ is a modification of Gentzen's system *NJ* from [6] (i.e. Prawitz's system from [11]) and Zucker's system $\mathcal{N}$ from [14]. (The system $\mathcal{NE}$ is similar to the systems from [9] and [13].) The most important characteristic of the system $\mathcal{NE}$ is that elimination rules for all connectives and quantifiers are of the same form as the elimination rules of $\vee$ and $\exists$ in Gentzen's system *NJ*. In [5] the definition of maximum cuts in derivations of the system $\mathcal{SE}$ was presented (which is in fact the definition of maximum cuts in derivations of the system $\mathcal{S}$). The following part of the connection of the kind (**) above was proved: the image of a sequent derivation without maximum cuts from the system $\mathcal{SE}$ is a normal derivation in the system $\mathcal{NE}$.

In this paper the second part of that connection will be presented: if a normal derivation of the system $\mathcal{NE}$ is the image of a sequent derivation from the system $\mathcal{SE}$, then that sequent derivation does not have maximum cuts.

In Section 2 we will briefly present connections between cut-free derivations and normal derivations from the system of sequents and natural deduction system (i.e. the connections of the kind (*)) from [3], [10] and [14]. In Section 3 the definition of maximum cuts in derivations of the system $\mathcal{S}$ from [4] will be repeated. In the first part of Section 4 the connection between derivations without maximum cuts of the system $\mathcal{S}$ and normal derivations of the system $\mathcal{N}$ from [4] (i.e. the connection of the kind (**)) will be presented. Finally, in the second part of Section 4 the system $\mathcal{SE}$ and the system $\mathcal{NE}$ from [5] will be considered and the property of the kind (**) will be shown for their derivations.

2. CUT-FREE DERIVATIONS AND NORMAL DERIVATIONS

In [14] Zucker defined a system of sequents, the system $\mathcal{S}$, and a natural deduction system, the system $\mathcal{N}$, which cover full intuitionistic predicate logic. The systems $\mathcal{S}$ and $\mathcal{N}$ are very similar to Gentzen's systems *LJ* and *NJ*, respectively. We present only Zucker's system $\mathcal{S}$. (His system $\mathcal{N}$ is a standard natural deduction system, i.e. Gentzen's system *NJ* with explicit contraction (see Section 2.3 in [14] for details).)

The system of sequents $\mathcal{S}$

A sequent of the system $\mathcal{S}$ has the form $\Gamma \to A$, where Γ is a finite set of indexed formulae and A is one unindexed formula. We only repeat the following about indices of formulae from [14] (for all other definitions see [14]): a finite non-empty sequence of natural numbers will be called a *symbol*, and will be denoted by $\sigma, \tau, \ldots$; a finite non-empty set of symbols will be called an *index*, and will be denoted by $\alpha, \beta, \ldots$ An index consisting of one symbol σ, $\{\sigma\}$, will be denoted just by σ. For any number i, the index $\{i\}$ (containing the single symbol i of length 1) will be called an *unary index*, and will be denoted just by i. There are two operations on indices: the *union* of two indices α and β, $\alpha \cup \beta$, is again an index and it is simply a set-theoretical union; and the *product* of α and β is $\alpha \times \beta =_{df} \{\sigma * \tau : \sigma \in \alpha, \tau \in \beta\}$, where $*$ is the concatenation of sequences.

Postulates for the system $\mathcal{S}$.

Initial sequents

logical initial sequents (i.e. i-sequents): $A_i \to A$.

$\bot$-*initial sequents (i.e.* $\bot$*-sequents):* $\bot_i \to P$, where P is any atomic formula different from $\bot$.

Inference rules

structural rules

$$\text{(contraction)}\ \frac{A_\alpha, A_\beta, \Gamma \to C}{A_{\alpha\cup\beta}, \Gamma \to C}$$

$$\text{(cut)}\ \frac{\Gamma \to A \qquad A_\alpha, \Delta \to C}{\Gamma_{\times\alpha}, \Delta \to C}$$

operational rules (i.e. rules for connecives)

left rules

$(\supset L)\ \dfrac{\Gamma \to A \qquad B_\beta, \Delta \to C}{\Gamma_{\times\beta}, A \supset B_\beta, \Delta \to C}$

$(\wedge L_1)\ \dfrac{A_\alpha, \Gamma \to C}{A \wedge B_\alpha, \Gamma \to C} \quad (\wedge L_2)\dfrac{B_\alpha, \Gamma \to C}{A \wedge B_\alpha, \Gamma \to C}$

$(\vee L)\ \dfrac{(A_\alpha), \Gamma \to C \qquad (B_\beta), \Delta \to C}{A \vee B_i, \Gamma, \Delta \to C}$

$(\forall L)\ \dfrac{Ft_\alpha, \Gamma \to C}{\forall x F x_\alpha, \Gamma \to C}$

$(\exists L)\ \dfrac{(F\mathrm{a}_\alpha), \Gamma \to C}{\exists x F x_i, \Gamma \to C}$

right rules

$(\supset R)\ \dfrac{(A_\alpha), \Gamma \to B}{\Gamma \to A \supset B}$

$(\wedge R)\ \dfrac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A \wedge B}$

$(\vee R_1)\ \dfrac{\Gamma \to A}{\Gamma \to A \vee B} \quad (\vee R_2)\dfrac{\Gamma \to B}{\Gamma \to A \vee B}$

$(\forall R)\ \dfrac{\Gamma \to F\mathrm{a}}{\Gamma \to \forall x F x}$

$(\exists R)\ \dfrac{\Gamma \to Ft}{\Gamma \to \exists x F x}$

The unary indices i from the initial sequents and the lower sequents in the left rules ($\vee$L) and ($\exists$L) are called *initial indices* (as Zucker's unary indices, see 2.2.1. in [14]), and they have to satisfy the *restrictions on indices*: in any derivation, all initial indices have to be distinct. In the rules ($\forall$R) and ($\exists$L) the variable a has to satisfy the well-known *restrictions on variables* (see 2.3.8.(b) in [14]). The notation $(C^c), \Theta \to D$, which is used in the rules ($\supset$R), ($\vee$L) and ($\exists$L), is interpreted as $C^c, \Theta \to D$, if $c \neq \emptyset$, and $\Theta \to D$, if $c = \emptyset$ (and hence not strictly an index, by our definition, see 2.2.8.(b) in [14]). So, $(C^c), \Theta \to D$ denotes either the sequent $C^c, \Theta \to D$ or the sequent $\Theta \to D$. The new formula explicitly shown in the lower sequent of an operational rule is the *principal formula*, and its subformulae from the upper sequents are the *side formulae* of that rule. The formula $A_{\alpha\cup\beta}$ is the *principal formula*, and A_α and A_β are the *side formulae* of the contraction. The formulae A and A_α from the upper sequents of the cut are the *cut formulae.* In any rule, formulae which are not side, principal or cut formulae, are *passive formulae* of that rule.

$\mathcal{D}, \mathcal{F}, \mathcal{D}', \mathcal{D}_0...$ will denote derivations in the system $\mathcal{S}$. $\begin{matrix}\mathcal{D}\\ \Gamma \to A\end{matrix}$ will denote a derivation $\mathcal{D}$ with the end sequent $\Gamma \to A$. All formulae making up sequents in a derivation $\mathcal{D}$ of the system $\mathcal{S}$ will be called *d-formulae of* the *derivation* $\mathcal{D}$.

2.1. Cut-free derivations of $\mathcal{S}^-$ and normal derivations of $\mathcal{N}^-$

In [14] Zucker defined the map φ from the set of derivations of the system $\mathcal{S}$ to the set of derivations of the system $\mathcal{N}$. In the system $\mathcal{S}$ derivations without cuts, i.e. cut-free derivations, were considered. In the system $\mathcal{N}$ maximum segments of derivations were Prawitz's maximum segments of natural deduction derivations (see [11] p. 49). Normal derivations of $\mathcal{N}$ were defined in an usual way, as derivations without maximum segments. In [14] the cut-elimination procedure in the system $\mathcal{S}$ and the normalization procedure in the system $\mathcal{N}$ consist of standard, i. e. Gentzen's ([6]) and Prawitz's ([11]) reductions, respectively. For these reductions Zucker solved the problem of the connection between reductions of the cut-elimination procedure in the system $\mathcal{S}^-$ and reductions of the normalization procedure in the system $\mathcal{N}^-$, where the systems $\mathcal{S}^-$ and $\mathcal{N}^-$ are the parts of the systems $\mathcal{S}$ and $\mathcal{N}$ which cover $(\wedge, \supset, \forall, \bot)$-fragment of intuitionistic logic. So, Zucker made the following connection between cut-free derivations of the system $\mathcal{S}^-$ and normal derivations of the system $\mathcal{N}^-$ (see Theorem 3 and Theorem 4 from Section 5 in [14]):

Theorem 1. *If $\mathcal{D}$ is a cut-free derivation from the system $\mathcal{S}^-$, then the derivation $\varphi\mathcal{D}$ is a normal derivation in the system $\mathcal{N}^-$.*

Theorem 2. *Let $\mathcal{D}$ be a derivation from the system $\mathcal{S}^-$. If the derivation $\varphi\mathcal{D}$ is a normal derivation in the system $\mathcal{N}^-$, then there is a cut-free derivation $\mathcal{D}_0$ such that the derivations $\mathcal{D}$ and $\mathcal{D}_0$ are connected by some reductions of the system $\mathcal{S}^-$.*

2.2. Normal derivations of $H_{\lambda L}$ and normal derivations of H_λ

In [10] Pottinger defined the system of sequents $H_{\lambda L}$ and the natural deduction system H_λ for intuitionistic propositional logic. He also defined the map $\mathcal{N}$ from the set of derivations of the system $H_{\lambda L}$ to the set of derivations of the system H_λ. The systems $H_{\lambda L}$ and H_λ are some kinds of λ-calculi. In the system $H_{\lambda L}$ derivations without cuts are called normal derivations. In derivations of the system H_λ maximum segments of derivations are Prawitz's maximum segments, and normal derivations are derivations without maximum segments. The set of reductions of the cut-elimination procedure in the system of sequents $H_{\lambda L}$ contains some not standard reductions, i.e. Zucker's less natural reductions (see the part 7.8.2(b) in [14]). Pottinger connected the reductions of that cut-elimination procedure in system the $H_{\lambda L}$ with Prawitz's reductions of the normalization procedure in the system H_{λ}. Moreover, he made the following connection (see Theorem 6.2 and Theorem 6.5 from Section 6 in [10]):

Theorem 3. *If $\mathcal{D}$ is a normal derivation from the system $H_{\lambda L}$, then the derivation $\mathcal{N}(\mathcal{D})$ is a normal derivation in the system H_{λ}.*

Theorem 4. *Let $\mathcal{D}$ be a derivation from the system $H_{\lambda L}$. If the derivation $\mathcal{N}(\mathcal{D})$ is a normal derivation in the system H_{λ}, then there is a normal derivation $\mathcal{D}'$ from the system $H_{\lambda L}$ such that $\mathcal{N}(\mathcal{D}) = \mathcal{N}(\mathcal{D}')$.*

2.3. Cut-free derivations of $\mathcal{S}$ and normal derivations of $\mathcal{N}$

In [3] Zucker's system $\mathcal{S}$ with its cut-free derivations and Zucker's system $\mathcal{N}$ with its normal derivations were considered. The map which connects derivations from $\mathcal{S}$ and $\mathcal{N}$ is also Zucker's map φ from [14]. In [3] a connection between reductions of the cut-elimination procedure in the full system $\mathcal{S}$ and reductions of the normalization procedure in the full system $\mathcal{N}$ was made. By using that connection Zucker's results for derivations of the systems $\mathcal{S}^-$ and $\mathcal{N}^-$ were extended for derivations of the systems $\mathcal{S}$ and $\mathcal{N}$. Namely, in [3] there is the following connection between cut-free derivations of the system $\mathcal{S}$ and normal derivations of the system $\mathcal{N}$ (see Theorem 4 and Theorem 5 from Section 4 in [3]):

Theorem 5. *If $\mathcal{D}$ is a cut-free derivation from the system $\mathcal{S}$, then the derivation $\varphi\mathcal{D}$ is a normal derivation in the system $\mathcal{N}$.*

Theorem 6. *Let $\mathcal{D}$ be a derivation from the system $\mathcal{S}$. If the derivation $\varphi\mathcal{D}$ is a normal derivation in the system $\mathcal{N}$, then there is a cut-free derivation $\mathcal{D}_0$ such that the derivations $\mathcal{D}$ and $\mathcal{D}_0$ are connected by some reductions of the system $\mathcal{S}$.*

3. MAXIMUM CUTS

In this section we will present the notions and results from [4] and Section 3 of [2]. We will consider Zucker's system $\mathcal{S}$ (which defined in our Section 2), and we will repeat the definition of maximum cuts in derivations of the system $\mathcal{S}$.

First we give an example of a maximum cut. In the derivation $\mathcal{E}$ from Example 1 below, the last cut, the cut **c4**, is a maximum cut. Roughly speaking, its left cut formula $B \supset A$ is connected with the rule $\supset\mathbf{R}$ (i.e. the introduction of $\supset$), and its right cut formula $B{\supset}A_{ilj}$ is connected with the rule $\supset\mathbf{L}$ (i.e the elimination of $\supset$).

Example 1 The derivation $\mathcal{E}$: (the formula A is $C{\lor}D$)

$$
\dfrac{
\dfrac{
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\dfrac{\dfrac{B_h{\to}B \quad C_g{\to}C}{B_{hg}, B{\supset}C_g{\to}C}}{B_{hg}, B{\supset}C_g{\to}C{\lor}D} \qquad A_f{\to}A
}{B_{hgf}, B{\supset}C_{gf}{\to}A}\,\text{c1}
}{B{\supset}C_{gf}{\to}\boldsymbol{B{\supset}A}}\,{\supset}\mathbf{R}
\qquad \boldsymbol{B{\supset}A_n{\to}B{\supset}A}
}{(B{\supset}C){\lor}F_k, \boldsymbol{B{\supset}A_n}{\to}\boldsymbol{B{\supset}A}}
\qquad
\dfrac{
\dfrac{\boldsymbol{B{\supset}A_i{\to}B{\supset}A} \qquad \boldsymbol{B{\supset}A_l{\to}B{\supset}A}}{\boldsymbol{B{\supset}A_{il}{\to}B{\supset}A}}\,\text{c2}
\qquad
\dfrac{\dfrac{B_m{\to}B \qquad A_j{\to}A}{B_{mj}, B{\supset}A_j{\to}A}\,{\supset}\mathbf{L}}{B_{mj}, \boldsymbol{B{\supset}A_j}{\to}A{\lor}E}
}{\boldsymbol{B{\supset}A}_{ilj}, B_{mj}{\to}A{\lor}E}\,\text{c3}
}{(B{\supset}C){\lor}F_{kilj}, \boldsymbol{B{\supset}A}_{nilj}, B_{mj}{\to}A{\lor}E}\,\mathbf{c4}
$$

To define maximum cuts of a derivation $\mathcal{D}$ we need to introduce several notions by which a precise connection between d-formulae in a derivation can be made. Some of these notions will be well-known notions from systems of sequents (see Note 3 below).

First we consider a formula A. One of its subformulae, a subformula C, will be called a *d-subformula C of A*, when the form of C and the place of its appearance in the formula A will be important. For example, the formula $A \equiv (C \supset D) \lor C$ has two different d-subformulae C. We note that the relation "... is a d-subformula of ..." is reflexive and transitive. A d-subformula of a formula A will be called a *proper d-subformula* when it is not the formula A itself. We also note that in a derivation, two d-formulae of the same form have the same d-subformulae which constitute them. (In the definition of a d-branch below we will use the following denotation: the indices of d-formulae will denote their place in a sequence of d-formulae where these formulae can or cannot be indexed formulae.)

Let $\mathcal{D}$ be a derivation, and A be a d-formula from $\mathcal{D}$. One *d-branch of* the *d-formula A in* the *derivation $\mathcal{D}$* will be a sequence of d-formulae $F_1, ..., F_n$, $n \geq 1$, where F_1 is that d-formula A, and for each i, $i \geq 1$ if F_i is

(i) either a passive formula in the lower sequent of a rule, or the principal formula of a contraction, then F_{i+1} is the corresponding passive formula from one of the upper sequents of that rule or one of the side formulae from the upper sequent of that contraction, respectively;

(ii) a principal formula in the lower sequent of an operational rule, then F_{i+1} is one of the side formulae (if they exist) from the upper sequents of the rule (which need not be on the same side of $\rightarrow$ as F_i);

(iii) a d-formula from an initial sequent, or the principal formula of a rule which does not have a side formula, then $i = n$.

Note 1. Our notion of a d-branch is very similar to the notion of the path in a derivation from natural deduction (see [11] p. 52).

In a derivation $\mathcal{D}$ the d-branch b of a d-formula A which is not a part of d-branches of any other d-formula from $\mathcal{D}$ will be called a *long d-branch* of that *d-formula A*.

Note 2. If in a derivation $\mathcal{D}$ the d-branch b is a long d-branch of a d-formula A, then the d-formula A is either a cut formula or a formula from the end sequent of the derivation $\mathcal{D}$.

In a derivation $\mathcal{D}$ for a d-branch b of a d-formula A we define a *branch of* the *d-formula A in $\mathcal{D}$* as the sequence of consecutive d-formulae from b (equal to A) whose first formula is the first formula of b, the d-formula A, and its last formula is a d-formula from b such that the next d-formula from b (if it exists) is different from A.

Note 3. All branches of a d-formula in a derivation form Gentzen's cluster (see [7] p. 267) of that d-formula in the derivation.

In Example 1 the left cut formula of the cut **c4** has the d-branch b_{l1} (which is its branch): $B \supset A$ (the left cut formula of the cut **c4** itself), $B \supset A$ (from $B \supset A_n \rightarrow$

$B \supset A$); and the branch b_{l2}: $B \supset A$ (the left cut formula of the cut **c4** itself), $B \supset A$ (the principal formula of $\supset$**R**). On the other hand, the right cut formula of the cut **c4** has the d-branch b_r (which is its branch): $B \supset A_{ilj}$, $B \supset A_{il}$, $B \supset A_i$. The branch b_{l2} connects the left cut formula of the cut **c4** with the rule $\supset$**R**, but the d-branch b_r does not connect the right cut formula of the cut **c4** with the rule $\supset$**L**. To make that connection we need to define the notion of the o-tree of a d-formula. In Example 1 the sequences of the bold emphasized formulae are the o-trees of the left and right cut formula of the cut **c4**. The o-tree $tr_r : t_1t_2t_3t_4t_5$ of the d-formula $B \supset A_{ilj}$ consists of the following parts: t_1 is b_r; t_2 is the inverted (i.e. written in the inverse order) long d-branch of the left cut formula $B \supset A$ of the cut c2, which is that d-formula itself; t_3 is the d-branch of the right cut formula $B \supset A_l$ of the cut c2, which is that d-formula itself; t_4 is the inverted long d-branch of the d-formula $B \supset A$ from $B \supset A_{il} \rightarrow B \supset A$ which consists of that d-formula and $B \supset A$ from $B \supset A_l \rightarrow B \supset A$; t_5 is the right cut formula $B \supset A_j$ of the cut c3. On the other hand, the left cut formula of the cut **c4** has two o-trees: tr_{l1} and tr_{l2}. The o-tree tr_{l1} is $t_1^{l1}t_2^{l1}$, where t_1^{l1} is b_{l1} and t_2^{l1} is the inverted long d-branch of the d-formula $B \supset A_{nilj}$:$B \supset A_n, B \supset A_n, B \supset A_{nilj}$. The o-tree tr_{l2} is the branch b_{l2}. Roughly speaking, in a derivation one o-tree of a d-formula C will consist of d-branches and inverted long d-branches of some d-formulae, alternately. The first part of an o-tree of a d-formula C will be a branch of that d-formula C. The next parts (if they exist), which make that o-tree, will be the d-branches of cut formulae and inverted long d-branches of cut formulae, alternately. The last part of that o-tree can be: the branch of the d-formula C which ends with the principal formula of an operational rule (see tr_{l2} above); a cut formula (see tr_r above); the inverted long d-branch of a d-formula from the end sequent of the derivation (see tr_{l1} above); or a d-formula from an initial sequent.

Now we define the notion of an o-tree of a d-formula.

First, for a d-branch $b : F_1...F_n$ of a d-formula A and its d-subformula C we define the following notions: (i) the sequence of d-formulae $b^{-1} : F_n...F_1$; (ii) the *d-branch b is* a *part of* C when F_n is a proper d-subformula of C; (iii) C *is* a *part of* the *d-branch* b when C is a d-subformula of F_n.

Let A be a d-formula from a derivation $\mathcal{D}$. An *o-tree of* the *d-formula* A *in* the *derivation* $\mathcal{D}$ (a $\mathcal{D}$*-tree of* A) will be a sequence $t_1...t_n$ $(n \geq 1)$, where t_1 is a branch of the d-formula A in $\mathcal{D}$, and t_i, $i > 1$, are some sequences of d-formulae from $\mathcal{D}$ which are made in the following way.

– If the last d-formula of t_1 is a principal formula of an operational rule, then $n = 1$.

– If the last d-formula of t_1 belongs to an initial sequent, then $n>1$ and for each k, $k\geq 1$:

If the last d-formula of t_{2k-1} is
(i) one d-formula of an i-sequent and C_m is other d-formula of that i-sequent, then t_{2k} is b^{-1}, where $b : C_1...C_m$ is a long d-branch which ends in C_m;
(ii) a d-formula from a $\bot$-sequent, then t_{2k} is the other d-formula from that $\bot$-sequent and n is $2k$.

If the last d-formula of t_{2k} is
(i) a d-formula from the end sequent of $\mathcal{D}$, then n is $2k$;
(ii) the d-formula C_1, which is a cut formula of a cut whose other cut formula is C (C_1 and C have the same form), then t_{2k+1} can be

(a) only the d-formula C, when there is a d-branch of C which is a part of A and $n = 2k + 1$;

(b) a d-branch of C which ends in an initial sequent and whose part is A (if it exists);

(c) one empty sequence, i.e. $n = 2k$, and t_{2k} has to be changed, it becomes only its first d-formula, otherwise.

In a derivation $\mathcal{D}$ an o-tree $tr : t_1...t_n$ of a d-formula A is *solid* if n is an even number, otherwise the o-tree tr is *not solid.*

In Example 1 for the o-trees tr_r, tr_{l1} and tr_{l2} we have the following. The o-tree tr_r is a not solid o-tree of the right cut formula of the cut **c4**, the d-formula $B\supset A_{ilj}$; the o-tree tr_{l1} is a solid o-tree of the left cut formula of the cut **c4**, the d-formula $B\supset A$, and the o-tree tr_{l2} is a not solid o-tree of that d-formula.

Lemma 1. *Let A be a d-formula in a derivation $\mathcal{D}$ and tr:$t_1...t_n$ be an o-tree of that d-formula A.*

(1) n is an even number iff the last d-formula of tr belongs to the end sequent of the derivation $\mathcal{D}$ or an initial sequent.

(2) n is an odd number iff the last d-formula of tr is either a principal formula of an operational rule or a cut formula whose one d-branch contains the principal formula A of an operational rule.

Proof. See the proof of Lemma 1 from Section 3 in [2]. □

All possible o-trees of a d-formula A in a derivation $\mathcal{D}$ form the *origin of* the *d-formula A in* the *derivation $\mathcal{D}$*. A *d-formula A has* the *safe origin in* a *derivation* $\mathcal{D}$ if all its o-trees are solid; otherwise that d-formula A has no safe origin in that derivation.

Lemma 2. *A d-formula A has the safe origin in a derivation $\mathcal{D}$ iff the last d-formula of each o-tree of A in the derivation $\mathcal{D}$ belongs to either the end sequent of the derivation $\mathcal{D}$ or an initial sequent.*

Proof. See the proof of Lemma 2 from Section 3 in [2]. □

Now we can define the notion of a maximum cut of a derivation.

Let $\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \Gamma \to A & A_\alpha, \Delta \to D\end{matrix}}{\Gamma_{\times\alpha}, \Delta \to D}$ cut be a subderivation of a derivation $\mathcal{D}$. The cut, which is the last rule of that subderivation, will be called a *maximum cut* (*m-cut*) *of* the *derivation $\mathcal{D}$* if neither the d-formula A from $\Gamma \to A$ nor the d-formula A_α from $A_\alpha, \Delta \to D$ has safe origin in the derivation $\mathcal{D}$. Otherwise, that cut will be called a *no-maximum cut of* the *derivation $\mathcal{D}$*.

In Example 1 the cuts c2, c3 and **c4** are maximum cuts and the cut c1 is a no-maximum cut of the derivation $\mathcal{E}$.

4. DERIVATIONS WITHOUT MAXIMUM CUTS AND NORMAL DERIVATIONS

4.1. The systems $\mathcal{S}$ and $\mathcal{N}$

In [4] Zucker's systems $\mathcal{S}$ and $\mathcal{N}$ and his map φ, which connects derivations of these systems, were considered. The definition of maximum cuts of derivations in the system $\mathcal{S}$ (which is repeated in our Section 3 above) was presented. In the system $\mathcal{N}$ Prawitz's definition of maximum segments was used, and normal derivations were defined in an usual way, as derivations without maximum segments. It was shown the following connection between derivations without maximum cuts of the system $\mathcal{S}$ and normal derivations of the system $\mathcal{N}$ (see Theorem 1 and Theorem 2 from Section 4.3 in [4]):

Theorem 7. *If a derivation $\mathcal{D}$ is a derivation without maximum cuts in the system $\mathcal{S}$, then the derivation $\varphi\mathcal{D}$ is a normal derivation in the system $\mathcal{N}$.*

Theorem 8. *Let $\mathcal{D}$ be a derivation in the system $\mathcal{S}$. If the derivation $\varphi\mathcal{D}$ is a normal derivation in the system $\mathcal{N}$, then $\mathcal{D}$ is a derivation without maximum cuts in the system $\mathcal{S}$.*

4.2. The systems $\mathcal{SE}$ and $\mathcal{NE}$

In [1] a new solution of the problems of connections between reductions of cut-elimination and normalization procedure was presented. The system of sequents $\mathcal{SE}$, which is the system $\delta\mathcal{E}$ with upper indices from [1] (see Section 2.1 from [1] for details), and a new natural deduction system, the system $\mathcal{NE}$, were considered.

We present only the system $\mathcal{NE}$ from [1]. (The system $\mathcal{SE}$ is the system $\mathcal{S}$ from our Section 2 with upper indices i.e. the system $\delta\mathcal{E}$ with only upper indices from [1].)

The natural deduction system $\mathcal{NE}$

Postulates in the system $\mathcal{NE}$ (see Section 2.3 from [1] for details).

Trivial derivation of A from A itself, A or A^i, where i is any unary index.

Substitution. From $\begin{matrix}\Delta\\ \pi_1\\ A\end{matrix}$ and $\begin{matrix}\Gamma, A^a\\ \pi_2\\ C\end{matrix}$ we define a derivation $\Gamma,\ \begin{matrix}\Delta^{\times a}\\ \pi_1\\ (A^a).\\ \pi_2\\ C.\end{matrix}$

Contraction: From $\begin{matrix}\Gamma, A^a, A^b\\ \pi\\ C\end{matrix}$ we make $\begin{matrix}\Gamma, A^{a*}, A^{b*}\\ \pi\\ C\end{matrix}$, where $*$ means that A^a and A^b are contracted.

Logical inference rules

introduction rules

$$\frac{\begin{matrix}[A^a]\\ \pi\\ B\end{matrix}}{A \supset B}\ (\supset I\mathcal{E})$$

$$\frac{A \quad B}{A \wedge B}\ (\wedge I\mathcal{E})$$

$$\frac{A}{A \vee B}\ (\vee I\mathcal{E}_1) \qquad \frac{B}{A \vee B}\ (\vee I\mathcal{E}_2)$$

$$\frac{F\mathrm{a}}{\forall x F x}\ (\forall I\mathcal{E})$$

$$\frac{F\mathrm{t}}{\exists x F x}\ (\exists I\mathcal{E})$$

elimination rules

$$\frac{\begin{matrix}\pi_1\\ A \supset B\end{matrix} \quad \begin{matrix}\pi_2\\ A\end{matrix} \quad \begin{matrix}[B^b]\\ \pi_3\\ C\end{matrix}}{C}\ (\supset E\mathcal{E})$$

$$\frac{\begin{matrix}\pi_1\\ A \wedge B\end{matrix} \quad \begin{matrix}[A^a]\\ \pi_2\\ C\end{matrix}}{C}\ (\wedge E\mathcal{E}_1) \qquad \frac{\begin{matrix}\pi_1\\ A \wedge B\end{matrix} \quad \begin{matrix}[B^b]\\ \pi_2\\ C\end{matrix}}{C}\ (\wedge E\mathcal{E}_2)$$

$$\frac{\begin{matrix}\pi_1\\ A \vee B\end{matrix} \quad \begin{matrix}[A^a]\\ \pi_2\\ C\end{matrix} \quad \begin{matrix}[B^b]\\ \pi_3\\ C\end{matrix}}{C}\ (\vee E\mathcal{E})$$

$$\frac{\begin{matrix}\pi_1\\ \forall x F x\end{matrix} \quad \begin{matrix}[F\mathrm{t}^a]\\ \pi_2\\ C\end{matrix}}{C}\ (\forall E\mathcal{E})$$

$$\frac{\begin{matrix}\pi_1\\ \exists x F x\end{matrix} \quad \begin{matrix}[F\mathrm{a}^c]\\ \pi_2\\ C\end{matrix}}{C}\ (\exists E\mathcal{E})$$

$\bot-$*rule*: $\dfrac{\bot}{P}\ (\bot)$, P is an atomic formula different from $\bot$.

In the rules $(\forall I\mathcal{E})$ and $(\exists E\mathcal{E})$ the variable a has to satisfy the well-known *restrictions on variables.* In each of the rules $(\supset I\mathcal{E})$, $(\supset E\mathcal{E})$, $(\wedge E\mathcal{E}_1)$, $(\wedge E\mathcal{E}_2)$, $(\vee E\mathcal{E})$, $(\forall E\mathcal{E})$ and $(\exists E\mathcal{E})$ in the brackets [ ] there is the assumption class which is *discharged by that rule.*

Introduction rules of the system $\mathcal{NE}$ are introduction rules of Gentzen's natural deduction system *NJ*. However, in the system $\mathcal{NE}$ elimination rules of all connectives and quantifiers are of the same form as the elimination rules of $\vee$ and $\exists$ in the system *NJ*. Maximum segments of derivations in the system $\mathcal{NE}$ are Prawitz's maximum segments (see [11] p. 49) and some new maximum segments which are made by the elimination rules of $\wedge$, $\supset$ and $\forall$ (for details see Section 5.3 in [1]). Normal derivations in the system $\mathcal{NE}$ are defined in an usual way, as derivations without maximum segments. In [1] it was shown that the normalization procedure in the system $\mathcal{NE}$ has the reductions corresponding to standard reductions of the cut-elimination procedure in the system $\mathcal{SE}$.

In [5] connections between derivations without maximum cuts from the system $\mathcal{SE}$ and normal derivations from the system $\mathcal{NE}$ were considered. To connect derivations from $\mathcal{SE}$ and $\mathcal{NE}$ the map ψ from [1] was used. Maximum cuts in derivations from the system $\mathcal{SE}$ were defined in the same way as maximum cuts in derivations from the system $\mathcal{S}$ in Section 3 above. In [5] the following property (i. e. the first part of the property (**) from our Introduction) was proved (see Theorem 3.1 in [5]):

Theorem 9. *If $\mathcal{D}$ is a derivation without maximum cuts in the system $\mathcal{SE}$, then the derivation $\psi\mathcal{D}$ is a normal derivation in the system $\mathcal{NE}$.*

By using our Lemma 2 from Section 3, Lemma 3.7 from [5], our Theorem 9 and Theorem 2 from Section IV§2 in [11] the second part of the property (**) (from our Introduction) for the systems $\mathcal{NE}$ and $\mathcal{SE}$, i.e. the following theorem, can be proved.

Theorem 10. *Let $\mathcal{D}$ be a derivation in the system $\mathcal{SE}$. If the derivation $\psi\mathcal{D}$ is a normal derivation in the system $\mathcal{NE}$, then $\mathcal{D}$ is a derivation without maximum cuts in the system $\mathcal{SE}$.*

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