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AN EXTENSION OF THE PROBABILITY LOGIC LPP_2

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Abstract. In this paper we present a probability logic which allows Boolean combinations of formulas of the form of $a_1 w(\varphi_1) + \cdots + a_k w(\varphi_k) \geq c$, where $\varphi_1, \ldots, \varphi_k$ are propositional formulas, $a_1, \ldots, a_k, c$ are rational numbers. The logic with such syntax was introduced in [1]. We present axiomatic system along with the ideas from [2, 3] and prove extended completeness, instead of simple completeness shown in [1].

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1. INTRODUCTION

In the past two decades, formalisms for representing uncertain information and knowledge and reasoning about them are the subject of increasing interest in many fields od science, starting with theoretical computer science, through artificial intelligence as first consumer of the research results in that field, distributed systems, cryptography etc. This interest is triggered by the fact that modeling and solving real-world problems often require dealing with a incomplete, inconsistent or vague information. One type of the approaches to representing uncertainty and reasoning about it uses formal frameworks based on probabilistic logic formalisms. In the literature two kinds of probabilistic logics can be found:

- the first one [1] allows linear inequalities involving probabilities (called *weighted formulas*) of the form $a_1 w(\varphi_1) + \cdots + a_k w(\varphi_k) \geq c$, where $\varphi_1, \ldots, \varphi_k$ are propositional formulas, $a_1, \ldots, a_k, c$ are integers,
- in the second approach [6, 2, 3] a countable list of probability operators of the form $P_{\geq s}$, is considered, with the intended meaning of $P_{\geq s}A$"the probability of A is at least s".

The authors in [1] failed to provide a complete axiomatization for the language without linear inequalities. Thus they introduced linear combinations, used their properties, propose a finite axiomatic system and obtained simple completeness ("Every valid formula is provable" or "Every consistent formula has a model"). However, since the compactness theorem ("A set is satisfiable if every its finite subset is satisfiable") does not hold for such kind of probability logics (consider an arbitrary classical propositional formula A and the set $T = \{\neg P_{>0}A\} \cup \{P_{<\frac{1}{n}}A | n$ is a positive integer$\}$, although every finite subset of T is satisfiable, the set T is not), extended completeness ("Every consistent set of formulas has a model") cannot be proved for a finite axiomatic system. In the second approach ([3]), the authors added an infinitary rule in their system, which resulted in a proof of extended completeness for the logic LPP_2 with the probability operators (but without linear inequalities). This paper considers the logic introduced in [1], but the corresponding axiomatic system includes an inference rule similar to the infinitary rule form [3], which results in a proof of extended completeness.

The rest of the paper is organized as follows. In Section 2 and 3 syntax and semantics of the logic are given. The axiomatic system is introduced in Section 4. In Section 5 soundness and completeness of the given logic are proved. Finally, we give our conclusion in Section 6.

2. SYNTAX

We introduce an extension of the probability logic LPP_2^{ext} which is similar to the logic presented in [1]. More precisely, the syntax is the same, thus expressivity of these two logics is the same, but there some differences between the corresponding axiomatic systems. These differences allow as to prove extended completeness, instead of simple completeness shown in [1]. In this section we describe the syntax of our logic.

The probability language of LPP_2^{ext} consists of:

- the denumerable set $\phi = \{p_1, p_2, \ldots\}$ of primitive propositions,
- the classical propositional operators $\neg$ and $\wedge$ and
- the symbol P^*.

The set of formulas consists of:

- the set of all classical propositional formulas denoted by For_C (the formulas from the set For_C will be denoted by $A, B, C, \ldots$),
- the set of weighted formulas denoted by For_W (the formulas from the set For_W will be denoted by $\alpha, \beta, \gamma, \ldots$), which are built as follows:
  - a primitive weighted term is an expression of the form $P^*(A)$, where A is a propositional formula,
  - a weighted term is an expression of the form $a_1P^*(A_1) + a_2P^*(A_2) + \cdots + a_kP^*(A_k)$, where $a_1, a_2, \ldots, a_k \in \mathbf{Q}$ and $k \geq 1$,
  - a basic weighted formulas are expressions of the form of $t \geq s$, where t is a weighted term and $s \in \mathbf{Q}$,
  - if α and β are weighted formulas then $\neg\alpha$ and $\alpha \wedge \beta$ are weighted formulas.

The usual abbreviations for the classical connectives $\alpha \vee \beta =_{def} \neg(\neg\alpha \wedge \neg\beta)$, $\alpha \rightarrow \beta =_{def} \neg\alpha \vee \beta$ and $\alpha \leftrightarrow \beta =_{def} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$ are used.

According to the properties of inequalities, we use the abbreviations:

- $t < s =_{def} \neg(t \geq s)$,
- $t \leq s =_{def} -t \geq -s$,
- $t > s =_{def} \neg(t \leq s)$,
- $t = s =_{def} (t \geq s) \wedge \neg(t > s)$.

The logic defined in the given way can be used to reason about probabilities. For example, the sentence *"probability of (event) p is less than $\frac{1}{3}$ and p is at least as twice probable as q"* can be expressed with $(3P^*(p) < 1) \wedge (P^*(p) - 2P^*(q) \geq 0)$. The formula $w(p) \geq \frac{1}{2}w(q)$ can be used to describe the assertion *"the conditional probability of p given q is at least 1/2"*([1]).

3. SEMANTICS

First we shell review some notions of probability theory. Let V be a non-empty set and a H be a class of subsets of V which contains V and is closed under complementation and finite union, i.e. H is an algebra. A finitely additive probability measure μ is defined as a function from H to the real interval $[0, 1]$ which satisfies: $\mu(V) = 1$ and $\mu(H_1 + H_2) = \mu(H_1) + \mu(H_2)$, for all disjoint sets $H_1, H_2 \in H$.

Definition 3.1. *An LPP_2^{ext} probability model is a structure $M = \langle W, v, H, \mu \rangle$ where:*

- *W is non-empty set of worlds,*
- *$v : W \times \phi \mapsto \{\top, \bot\}$ is propositional valuation, which associates every world with every primitive proposition valuating their association with the truth assignment,*
- *H is an algebra of subsets of W,*
- *$\mu : H \mapsto [0, 1]$ is finitely-additive probability measure over H (in difference with* [1] *where μ was defined as countable additive).*

If A is a classical propositional formula and $w \in W$, where $M = \langle W, v, H, \mu \rangle$ is an $LPP^{ext}_{2,Maes}$-model, then $w \models A$ and $v(w, A) = \top$ denote that A is satisfied under interpretation v.

Definition 3.2. *LPP^{ext}_2-model $M = \langle W, v, H, \mu \rangle$ is measurable if*

$$\{w \in W : w \models A\} \in H$$

holds for every propositional formula A.

The class of all measurable models will be denoted by $LPP^{ext}_{2,Maes}$.

Definition 3.3. *Let $M = \langle W, v, H, \mu \rangle$ be an $LPP^{ext}_{2,Maes}$-model. A formula Φ is satisfied in the model M ($M \models \Phi$) if the following holds:*

- *if Φ is a proposition formula A, $M \models A$ iff $v(w, A) = \top$, for every world $w \in W$,*
- *if Φ is a basic weighted formula, $M \models a_1 P^*(A_1) + \cdots + a_k P^*(A_k) \geq s$ iff $\sum_{i=1}^{k} a_i \mu(\{w : w \in W, w \models A_i\}) \geq s$,*
- *if Φ has the form $\neg\alpha$, where α is weighted formula, $M \models \neg\alpha$ iff $M \models \alpha$ does not hold,*
- *if Φ has the form $\alpha \wedge \beta$, where α and β are weighted formulas, $M \models \alpha \wedge \beta$ iff $M \models \alpha$ and $M \models \beta$.*

Definition 3.4. *A formula Φ is satisfiable if there is an $LPP^{ext}_{2,Maes}$-model M such that $M \models \Phi$. A formula Φ is valid if for every $LPP^{ext}_{2,Maes}$-model M holds $M \models \Phi$. A set of formulas is satisfiable if there is a model which satisfies every formula from the set.*

4. AXIOMATIC SYSTEM

The axiomatic system $Ax_{LPP^{ext}_2}$ for LPP^{ext}_2 contains the following axiom schemata:

1. All instances of classical propositional tautologies,

2. All probabilistic instances of valid formulas concerning linear inequalities
   (a) $x \geq x$,
   (b) $(a_1x_1 + \cdots + a_kx_k \geq c) \Leftrightarrow (a_1x_1 + \cdots + a_kx_k + 0x_{k+1} \geq c)$,
   (c) $(a_1x_1 + \cdots + a_kx_k \geq c) \Rightarrow (a_{j_1}x_{j_1} + \cdots + a_{j_k}x_{j_k} \geq c)$, where $j_1, \ldots j_k$ is permutation of $1, \ldots, k$,
   (d) $(a_1x_1 + \cdots + a_kx_k \geq c) \wedge (a'_1x_1 + \cdots + a'_kx_k \geq c') \Rightarrow ((a_1 + a'_1)x_1 + \cdots + (a_k + a'_k)x_k \geq (c + c'))$,
   (e) $(a_1x_1 + \cdots + a_kx_k \geq c) \Leftrightarrow (da_1x_1 + \cdots + da_kx_k \geq dc)$, if $d > 0$,
   (f) $(t \geq c) \vee (t \leq c)$, if t is a term,
   (g) $(t \geq c) \Rightarrow (t > d)$, if t is a term and $c > d$, i.e. $(t \leq c) \Rightarrow (t < d)$, if t is a term and $c < d$,
   (h) $(t > c) \Rightarrow (t \geq c)$, i.e. $(t < c) \Rightarrow (t \leq c)$ if t is a term,
3. All instances of formulas for probabilistic reasoning
   (a) $P^*(A) \geq 0$,
   (b) $P^*(A) \leq s \leftrightarrow P^*(\neg A) \geq 1 - s$,
   (c) $(P^*(A) \geq r \wedge P^*(B) \geq s \wedge P^*(\neg A \vee \neg B) \geq 1) \rightarrow$
   $P^*(A \vee B) \geq \min(1, r + s)$,
   (d) $(P^*(A) \leq r \wedge P^*(B) < s) \rightarrow P^*(A \vee B) < r + s, \; r + s \leq 1$,

and the inference rules:

1. From Φ and $\Phi \rightarrow \Psi$ infer Ψ,
2. For a propositional formula A, from A infer $P^*(A) \geq 1$,
3. For a weighted term t, from $\beta \rightarrow t \geq s - \frac{1}{k}$, for every integer $k \geq \frac{1}{s}$, infer $\beta \rightarrow t \geq s$.

Definition 4.1. *A formula Φ is a theorem in the axiomatic system $Ax_{LPP_2^{ext}}$ ($\vdash_{Ax_{LPP_2^{ext}}} \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is an axiom or it is derived from the preceding formulas by an inference rule. The sequence $\Phi_0, \Phi_1, \ldots, \Phi$ is a proof for Φ.*

Definition 4.2. *A formula Φ is deducible from a set of formulas $T (T \vdash_{Ax_{LPP_2^{ext}}} \Phi)$ if there exists an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every*

Φ_i is an axiom or belongs to T or it is derived from the preceding formulas by an inference rule. Sequence $\Phi_0, \Phi_1, \dots, \Phi$ is the deduction (inference) for Φ from set of formulas T.

Definition 4.3. *A set of formulas T is consistent if there are at least one propositional formula A and one weighted formula α that are not deducible from T, i.e. $T \vdash A$ and $T \vdash \alpha$ do not hold. Otherwise T is inconsistent.*

Definition 4.4. *A set of formulas T is said to be maximal consistent if it is consistent and the following holds:*

- *for every propositional formula A, if $T \vdash A$, then $A \in T$ and $P^*(A) \geq 1 \in T$, and*
- *for every formula α (which is not propositional), either $\alpha \in T$ or $\neg\alpha \in T$.*

5. SOUNDNESS AND COMPLETENESS

In this section we will prove that the given axiomatic system is sound and complete.

5.1. Soundness

Theorem 5.1. (Soundness theorem) *The axiomatic system $Ax_{LPP_2^{ext}}$ is sound with respect to $LPP_{2,Meas}^{ext}$ class of models.*

Proof. Let M be an arbitrary $LPP_{2,Meas}^{ext}$ model. Considering propositional formulas every world of M can be seen as a usual propositional model, hence all propositional tautologies are satisfied in every world. The axioms 2a-2h represent properties of linear inequalities and hold in every model. The axioms 3a-3d consider symbol P^*, i.e. measure properties, and hold in every model, also.

Let us assume that α and $\alpha \to \beta$ are valid formulas. Now, if β is not valid there exists a world w in a model M such that $w \not\models \beta$ and $w \models \alpha \to \beta$. It means that

$w \not\models \alpha$, which does not hold according to the assumptions. Hence, the inference rule 1 preserves validity.

Considering the inference rule 2 let us assume that A is valid propositional formula. Then $(\forall w \in W) w \models A$ holds. Since $\mu(W) = 1$, it follows that $w \models P^*(A) \geq 1$, i.e. $P^*(A) \geq 1$ is satisfied in every world of every model. Hence, the inference rule 2 preserves validity, also. Finally, the rule 3 preserves validity because of the properties of real numbers. □

5.2. Completeness

Theorem 5.2. (Deduction theorem) *If T is a set of formulas and $T \cup \{\Phi\} \vdash \Psi$, then $T \vdash \Phi \rightarrow \Psi$, where Φ and Ψ are either both propositional or both weighted formulas.*

Proof. We use transfinite induction on the length of the inference (deduction) for Ψ. If the length of inference is 1, formula Ψ is either axiom or $\Psi \in T$. As $\Psi \rightarrow (\Phi \rightarrow \Psi)$ is a tautology, we can apply the rule 1 and obtain that $T \vdash \Phi \rightarrow \Psi$.

Let us assume that the length of the inference for Ψ is $k > 1$. Again, formula Ψ can be an axiom or an element of T, thus, as we previously concluded, $T \vdash \Phi \rightarrow \Psi$ holds. On the other hand, formula Ψ can be obtained by an application of some inference rule on formulas conducted in previous steps of inference.

First, let us assume that Ψ is obtained from $T \cup \{\Phi\}$ applying the rule 1. In such case, it has two assumptions in form γ and $\gamma \rightarrow \Psi$. Proofs (inferences) for both of them are shorter than k. By the induction hypothesis, $T \vdash \Phi \rightarrow \gamma$ and $T \vdash \Phi \rightarrow (\gamma \rightarrow \Psi)$. As formula $(\Phi \rightarrow (\gamma \rightarrow \Psi)) \rightarrow ((\Phi \rightarrow \gamma) \rightarrow (\Phi \rightarrow \Psi))$ is tautology, double application of the inference rule 1 results in $T \vdash \Phi \rightarrow \Psi$.

Secondly, if Ψ is obtained from $T \cup \{\Phi\}$ applying the rule 2, i.e. $\Psi = P^*(A) \geq 1$, thus $T \cup \Phi \vdash A$. As Φ and Ψ are either both propositional or both weighted formulas and Ψ is weighted formula, then Φ is a weighted formula. Thus $T \vdash A$ and $T \vdash P^*(A) \geq 1$. Since $\vdash \Psi \rightarrow (\Phi \rightarrow \Psi)$ and $T \vdash \Psi$, obtaining by the rule 1 we have $T \vdash \Phi \rightarrow \Psi$.

Finally, let us assume that $\Psi = \gamma \rightarrow t \geq s$ is obtained by applying the rule 3:

$T, \Phi \vdash \gamma \to t \geq s - \frac{1}{k}$, for every integer $k \geq \frac{1}{s}$,

$T \vdash \Phi \to (\gamma \to t \geq s - \frac{1}{k})$, for every integer $k \geq \frac{1}{s}$, by the induction hypotheses,

$T \vdash (\Phi \wedge \gamma) \to t \geq s - \frac{1}{k}$, for every integer $k \geq \frac{1}{s}$,

$T \vdash (\Phi \wedge \gamma) \to t \geq s$, by the inference rule 3,

$T \vdash \Phi \to \Psi$. □

Now, we will give some theorems and properties of LPP_2^{ext}, that will be used in the rest of the paper.

Theorem 5.3. *The following holds in corresponding LPP_2^{ext} logic:*

1. $\vdash P^*(A) \geq s \to P^*(A \vee \bot) \geq s$,
2. $\vdash P^*(A \vee \bot) \geq s \to P^*(\neg\neg A) \geq s$,
3. $\vdash P^*(A) \geq s \to P^*(\neg\neg A) \geq s$,
4. $\vdash P^*(A \to B) \geq 1 \to (P^*(A) \geq s \to P^*(B) \geq s)$,
5. *If* $\vdash A \leftrightarrow B$ *then* $\vdash P^*(A) \geq s \leftrightarrow P^*(B) \geq s$,
6. $\vdash P^*(A) \geq r \to P^*(A) \geq s, r > s$.

The corresponding proof of this theorem can be found in [5].

Theorem 5.4. *Every consistent set of formulas T can be extended to a maximal consistent set of formulas.*

Proof. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of all weighted formulas in T and $\overline{T}^C$ set of all classical propositional consequences of T. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$ such that:

1. $T_0 = T \cup \overline{T}^C \cup \{P^*(A) \geq 1 : A \in \overline{T}^C\}$,
2. for every $i \geq 0$, if $T_i \cup \{\alpha_i\}$ is consistent, then $T_{i+1} = T_i \cup \{\alpha_i\}$, otherwise
3. if $T_i \cup \{\alpha_i\}$ is not consistent, then $T_{i+1} = T_i \cup \{\neg\alpha_i\}$,

4. If T_{i+1} is obtained by adding a formula of the form $\neg(\beta \to t \geq s)$, then for some $n \in \mathbf{N}$ formula $\beta \to \neg(t \geq s - \frac{1}{n})$, i.e. $\beta \to t < s - \frac{1}{n}$ is also added to T_{i+1}, so that T_{i+1} is consistent,
5. $\overline{T} = \cup_i T_i$.

The sets obtained by the steps 1 and 2 are obviously consistent. We will now check consistency of sets obtained by the steps 3 and 4 respectively. If $T_i \cup \{\alpha_i\} \vdash \bot$, then, by the deduction theorem, $T_i \vdash \neg\alpha_i$ holds. Thus, since T_i is consistent, so is $T_i \cup \{\neg\alpha_i\}$.

Considering the step 4, we assume that set $T_i \cup \{\beta \to t \geq s\}$ is not consistent. Therefore T_{i+1} is equal to consistent set $T_i \cup \{\neg(\beta \to t \geq s)\}$. If T_{i+1} could not be extended by the formula $\beta \to \neg t \geq s - \frac{1}{k}$, then the following would hold:

$T_i, \neg(\beta \to t \geq s), \beta \to \neg t \geq s - \frac{1}{k} \vdash \bot$, for every integer $k > \frac{1}{s}$, by the hypothesis,

$T_i, \neg(\beta \to t \geq s) \vdash \neg(\beta \to \neg t \geq s - \frac{1}{k})$ for every integer $k > \frac{1}{s}$, by the deduction theorem,

$T_i, \neg(\beta \to t \geq s) \vdash \beta \to t \geq s - \frac{1}{k}$ for every integer $k > \frac{1}{s}$, from 2., by the classical tautology $\neg(\alpha \to \gamma) \to (\alpha \to \neg\gamma)$,

$T_i, \neg(\beta \to t \geq s) \vdash \beta \to t \geq s$, by the inference rule 3,

$T_i \vdash \neg(\beta \to t \geq s) \to (\beta \to t \geq s)$, by the deduction theorem,

$T_i \vdash \beta \to t \geq s$.

Since $T_i \cup \{\beta \to t \geq s\}$ is not consistent, from $T_i \vdash \beta \to t \geq s$ follows that T_i is not consistent, a contradiction. Thus, the step 4 produces consistent sets.

Finally, let us show that the set $\overline{T}$ obtained by the step 5 is consistent. This can be concluded from the fact that it is deductively closed and does not contain all formulas, which will be proved in further text.

According to the step 1, $\overline{T}$ contains all it's classical propositional consequences and all formulas in the form $P^*(A) \geq 1$ for $\overline{T} \vdash A$, $A \in \overline{T}^C$. Since T is consistent, not all classical propositional formulas are deductable from it, so the same holds for the

$\overline{T}$. It also should be noted that, for every formula α, if $T_i \vdash \alpha$, then it must be $\alpha \in \overline{T}$. For if $\alpha = \alpha_k$ and $\alpha \notin \overline{T}$, then $T_{\max\{i,k\}+1} \vdash \alpha$ and $T_{\max\{i,k\}+1} \vdash \neg\alpha$, a contradiction.

Let $\overline{T} \vdash \alpha$. If the inference for α from $\overline{T}$ contains finite number of steps, then there exists $i \geq \mathbf{N}$ such that $T_i \vdash \alpha$, hence $\alpha \in \overline{T}$. Now, let us assume that the sequence of formulas $\beta_1, \beta_2, \ldots, \alpha$, which represents the proof for α from $\overline{T}$ is enumerable countable. In that case we prove that for every i holds the following: if β_i is obtained by an application of an inference rule and all its premises belong to $\overline{T}$, then $\beta_i \in \overline{T}$.

First, we assume that β_i is obtained by the inference rule 1 and its premises β_i^1 i β_i^2 belong to $\overline{T}$. In such case, there must exist $k \in \mathbf{N}$ such that $\beta_i^1, \beta_i^2 \in T_k$. Since $T_k \vdash \beta_i$, it must be $\beta_i \in \overline{T}$. Another possible way of obtaining β_i is by applying the inference rule 2. Considering construction of $\overline{T}$, it must be $T_k \vdash \beta_i$, for some $T_k \vdash \beta_i$, and $\beta_i \in \overline{T}$, also. If not, then $\alpha_k = \neg\beta_i$, $\alpha_k \in T_{k+1}$, and T_{k+1} would not be consistent.

Finally, $\beta_i = \beta \to t \geq s$ can be the result of application infinitely the inference rule 3 on formulas $\beta_i^1 = \beta \to t \geq s - \frac{1}{k}, \beta_i^2 = \beta \to t \geq s - \frac{1}{k+1}, \ldots$ which belong to $\overline{T}$. If $\beta \to t \geq s \notin \overline{T}$, according to the step 4 of construction, there exists $j \in \mathbf{N}$ such that $j > \frac{1}{s}$ and $\beta \to \neg t \geq s - \frac{1}{j} \in \overline{T}$. Let $l = \max\{k, j\}$. By the axioms 2g and 2h, $\beta \to t \geq s - \frac{1}{l} \in \overline{T}$ and $\beta \to \neg t \geq s - \frac{1}{l} \in \overline{T}$. Further, for some $m \in \mathbf{N}$ set T_m also contains these two formulas. Set $T_m \cup \{\beta\}$ is not consistent and $\beta \notin \overline{T}$. Therefore, there exists some $j \in \mathbf{N}$ such that $\neg\beta \in T_j$, $T_j \vdash \beta \to \perp$, $T_j \vdash \beta \to t \geq s$ and $\beta \to t \geq s \in \overline{T}$, a contradiction. It follows that set $\overline{T}$ is deductively closed. Set $\overline{T}$ does not contain all formulas. If it could, for some weighted formula, α and $\neg\alpha$ belong to $\overline{T}$, then there would exist some $i \in \mathbf{N}$ such that $\alpha, \neg\alpha \in T_i$, a contradiction.

Therefore, $\overline{T}$ is consistent since there exist at least one propositional formula A and at least one weighted formula α such that neither $\overline{T} \vdash A$ nor $\overline{T} \vdash \alpha$.

Finally, by construction steps 2 and 3, for every weighted formula $\alpha \in \overline{T}$ or $\neg\alpha \in \overline{T}$, so $\overline{T}$ is maximal consistent set. □

Theorem 5.5. *Let $\overline{T}$ be a maximal consistent set of formulas. Then the following holds:*

1. *If $\Phi \in \overline{T}$, then $\neg\Phi \notin \overline{T}$,*

2. *If* $\overline{T} \vdash \Phi$*, then* $\Phi \in \overline{T}$*, i.e.* $\overline{T}$ *is deductively closed,*
3. $\Phi \wedge \Psi \in \overline{T}$ *iff* $\Phi \in \overline{T}$ *and* $\Psi \in \overline{T}$*,*
4. *If* $\Phi \in \overline{T}$ *and* $\Phi \to \Psi \in \overline{T}$*, then* $\Psi \in \overline{T}$*,*
5. *All theorems belongs to set* $\overline{T}$*,*
6. *If* $t \geq s \in \overline{T}$ *i* $s \geq r$*, then* $t \geq r \in \overline{T}$*,*
7. *If* $r \in \mathbf{Q}$ *and* $r = \sup\{s : t \geq s \in \overline{T}\}$*, then* $t \geq r \in \overline{T}$*.*

Proof.

1. If for some formula Φ the set $\overline{T}$ contains both Φ and $\neg\Phi$, then:

   $\overline{T} \vdash \Phi$,
   $\overline{T} \vdash \neg\Phi$,
   $\overline{T} \vdash \Phi \wedge \neg\Phi$,

   would hold and set $\overline{T}$ is not consistent.

2. Let $\overline{T} \vdash \Phi$ and $\Phi \notin \overline{T}$. Then $\neg\Phi \in \overline{T}$. Similarly as before $\overline{T} \vdash \Phi \wedge \neg\Phi$ so $\overline{T}$ is not consistent.

3. For all formulas $\Phi, \Psi \in \overline{T}$, the following holds:

   $\overline{T} \vdash \Phi$,
   $\overline{T} \vdash \Psi$,
   $\overline{T} \vdash \Phi \wedge \Psi$,
   $\Phi \wedge \Psi \in \overline{T}$, since $\overline{T}$ is deductively closed.

   Conversely, if $\Phi \wedge \Psi \in \overline{T}$, then:

   $\overline{T} \vdash \Phi \wedge \Psi$,
   $\overline{T} \vdash (\Phi \wedge \Psi) \to \Phi$,
   $\overline{T} \vdash (\Phi \wedge \Psi) \to \Psi$,
   $\overline{T} \vdash \Phi$,
   $\overline{T} \vdash \Psi$,

   hence, $\Phi, \Psi \in \overline{T}$ according to $\overline{T}$ is deductively closed.

4. In the same manner as in the previous step:

$\overline{T} \vdash \Phi$,

$\overline{T} \vdash \Phi \rightarrow \Psi$,

$\overline{T} \vdash \Psi$,

$\Psi \in \overline{T}$, because $\overline{T}$ is deductively closed.

5. Again, because $\overline{T}$ is deductively closed, if Φ is theorem, then $\overline{T} \vdash \Phi$, so $\Phi \in \overline{T}$,

6. According to the step 5, $t \geq s \rightarrow t \geq r \in \overline{T}$ for $s \geq r$. If $t \geq s \in \overline{T}$, then $t \geq r \in \overline{T}$, by 4,

7. Let $r = \sup\{s : t \geq s \in \overline{T}\}$. By the inference rule 3, $\overline{T} \vdash t \geq r$. Since $\overline{T}$ is deductively closed set $t \geq r \in \overline{T}$. □

Let be the tuple $M = \langle W, v, H, \mu \rangle$ such that:

- W is the set of all classical propositional interpretations which satisfy the set $\overline{T}^C$, i.e. $W = \{w : w \models \overline{T}^C\}$,
- v is mapping which associates every world $w \in W$ with interpretation, i.e. $v(w)(p) = w(p)$ for all primitive propositions $p \in \phi$,
- H is the class of sets in the form $[A] = \{w \in W : w \models A\}$, for all propositional formulas A and
- for every $[A] \in H$, $\mu([A]) = \sup\{r : P^*(A) \geq r \in \overline{T}\}$.

In a following theorem we will show that the tuple M is an $LPP^{ext}_{2,\text{Meas}}$-model.

Theorem 5.6. *Let $M = \langle W, v, H, \mu \rangle$ be a tuple defined as above. Then the following holds:*

1. *If $[A] = [B]$, then $\mu([A]) = \mu([B])$*
2. $\mu([A]) \geq 0$.
3. $\mu(W) = 1$ *and* $\mu(\emptyset) = 0$.
4. $\mu([A]) = 1 - \mu([\neg A])$.
5. $\mu(\theta_1 \cup \theta_2) = \mu(\theta_1) + \mu(\theta_2)$, *for all disjoint sets* $\theta_1, \theta_2 \in \{[A] : A \text{ is propositional formula}\}$.

Proof.

1. It is enough to prove that $[A] \subset [B]$ implies $\mu([A]) \leq \mu([B])$. If $[A] \subset [B]$ then $\vdash \neg(A \wedge \neg B)$, i.e. $\vdash A \to B$, thus, by the inference rule 2, $\vdash P^*(A \to B) \geq 1$. Further, using the Theorem 5.3.4 and the inference rule 1 we get $\vdash P^*(A) \geq s \to P^*(B) \geq s$, therefore $\mu([A]) \leq \mu([B])$.
2. Since $P^*(A) \geq 0$ is an axiom, $\mu([A]) \geq 0$ holds.
3. First, $\mu(\emptyset) \geq 0$ holds, since $p \vee \neg p, P^*(p \vee \neg p) \geq 1 \in \overline{T}$ for every $p \in \phi$ and $w \models p \vee \neg p$ for every $w \in W$. It is obvious that $\mu(\emptyset) \geq 0$. On the other hand:
$$\begin{aligned} P^*(p \vee \neg p) \geq 1 &\Leftrightarrow P^*(p \vee \neg p) \geq 1 - 0 \\ &\Leftrightarrow P^*(\neg(p \vee \neg p)) \leq 0 \\ &\Leftrightarrow P^*(p \wedge \neg p) \leq 0 \\ &\Leftrightarrow \neg P^*(p \wedge \neg p) > 0. \end{aligned}$$
Finally, $\sup\{s : P^*(p \wedge \neg p) \geq s \in \overline{T}\} = 0$, by the axiom 2g.
4. Let us consider nontrivial cases where $[A] \neq W$ and $[A] \neq \emptyset$ (for the trivial ones statements follow from 3). Let $r = \mu([A]) = \sup\{s : P^*(A) > s \in \overline{T}\}$ and $r < 1$. Then $\neg(P^*(A) \geq r') \in \overline{T}$, i.e. $P^*(A) < r' \in \overline{T}$, for every $r' \in (r, 1]$. By the axiom 2h we get $P^*(A) \leq r' \in \overline{T}$, therefore $P^*(\neg A) \geq 1 - r' \in \overline{T}$. On the other hand, if there exists rational number $r'' \in [0, r)$ such that $P^*(\neg A) \geq 1 - r'' \in \overline{T}$, then $\neg P^*(A) > r'' \in \overline{T}$, i.e. $\neg P^*(A) \geq r \in \overline{T}$, a contradiction.
5. Let $\theta_1 = [A]$, $\theta_2 = [B]$, $[A] \cap [B] = \emptyset$, $\mu([A]) = r$ and $\mu([B]) = s$. From $[B] \subset [\neg A]$, by 1 and 4, follows $r + s \leq r + (1 - s) = 1$. Let $r > 0$ and $s > 0$. By the known properties of supremum and monotonicity of measure (Theorem 5.3.6), for all rational numbers $r' \in [0, r)$ and $s' \in [0, s)$ holds $P^*(A) \geq r'$, $P^*(B) \geq s' \in \overline{T}$. Further, $P^*(A \vee B) \geq r' + s' \in \overline{T}$, by the axiom 3c. Hence, $r + s \leq \sup\{t : P^*(A \vee B) \geq t \in \overline{T}\}$. If $r + s = 1$, then assertion trivially holds. Suppose that $r + s < 1$. If $r + s < t_0 = \sup\{t : P^*(A \vee B) \geq t \in \overline{T}\}$, then $P^*(A \vee B) \geq t' \in \overline{T}$, for every rational number $t' \in (r + s, t_0)$. Finally, for rational numbers $r'' > r$ i $s'' > s$ such that $\neg P^*(A) \geq r'', P^*(A) < r'' \in \overline{T}$, $\neg P^*(B) \geq s'', P^*(B) < s'' \in \overline{T}$ and $r'' + s'' = t'' \leq 1$, by the axiom 2h holds $P^*(A) \leq r'' \in \overline{T}$. By the axiom 3d, $P^*(A \vee B) < r'' + s'', \neg P^*(A \vee B) \geq r'' + s'', \neg P^*(A \vee B) \geq t' \in \overline{T}$, a contradiction. □

Theorem 5.7. (Extended completeness) *Every consistent set of formulas T has an $LPP^{ext}_{2,Meas}$-model.*

Proof. Let T be a consistent set of formulas. T can be extended to a maximal consistent set $\overline{T}$. For $\overline{T}$ we can define model M in the previously described way. By the induction on the complexity of formulas we can prove that for every formula Φ, $M \models \Phi$ iff $\Phi \in \overline{T}$.

If Φ is a primitive proposition, then assertion holds by the model definition.

Let Φ be a propositional formula A. If $A \in \overline{T}$, then $M \models A$, according to the model definition. Conversely, if $M \models A$, meaning that A is satisfied in all classical propositional interpretations of worlds from the model M, therefore, by the completeness of classical propositional logic, it must be $A \in \overline{T}^C$, as well as $A \in \overline{T}$.

Let $\Phi = a_1P^*(A_1) + a_2P^*(A_2) + \cdots + a_kP^*(A_k) \geq s$ be a basic weighted formula. The proof will be obtained by the induction on the number of primitive weighted terms in the formula.

Let $k = 1$. If $P^*(A) \geq s \in \overline{T}$, then $\sup\{r : P^*(A) \geq r \in \overline{T}\} = \mu([A]) \geq s$ and $M \models P^*(A) \geq s$. Conversely, assume that $M \models P^*(A) \geq s$, i.e. $\sup\{r : P^*(A) \geq r \in \overline{T}\} = \mu([A]) \geq s$. If $\mu([A]) > s$ and $P^*(A) \geq s \notin \overline{T}$, then considering steps 2 and 3 of construction of set $\overline{T}$ formulas $\neg(P^*(A) \geq s), \neg(P^*(A \geq s - \frac{1}{n})$ belong to $\overline{T}$, for some $n \in \mathbf{N}$. On the other hand, for every $r \in \mathbf{Q}, \mathbf{r} \leq \mu([\mathbf{A}])$ all formulas $P^*(a) \geq r$ belong to $\overline{T}$, a contradiction. So, $P^*(A) \geq s \in \overline{T}$. For $\mu([A]) = s$, by the Theorem 5.5.7, $P([A]) \geq s \in \overline{T}$.

Let us assume that the assertion holds for all formulas where number of containing primitive weighted terms is smaller than k. Let $\Phi = a_1P^*(A_1) + a_2P^*(A_2) + \cdots + a_kP^*(A_k) \geq s$, $a_k > 0$, and $\Phi \in \overline{T}$. Throughout the proof, $s_i = \mu([A_i]) = \sup\{t : P^*(A_i) \geq t \in \overline{T}\}$ for $i = 1, k$. For every rational number $r_k \geq s_k$:

$\sum_{i=1}^{k-1} a_iP^*(A_i) + a_kP^*(A_k) \geq s \in \overline{T}$,

$\sum_{i=1}^{k-1} a_iP^*(A_i) \geq s - a_kP^*(A_k) \in \overline{T}$, using axiom 2d,

$\sum_{i=1}^{k-1} a_iP^*(A_i) \geq s - a_kr_k \in \overline{T}$, using axioms 2g and 2h,

$M \models \sum_{i=1}^{k-1} a_iP^*(A_i) \geq s - a_kr_k$, by the induction hypothesis, i.e,

$\sum_{i=1}^{k-1} a_i\mu([A_i]) \geq s - a_k r_k$.

Since $\mu([A_k]) = s_k$, it follows:

$\sum_{i=1}^{k} a_i\mu([A_i]) \geq s - a_k r_k + a_k s_k$,

$\sum_{i=1}^{k} a_i\mu([A_i]) \geq s - a_k(r_k - s_k)$.

As r_k is arbitrary close to s_k it also follows that

$\sum_{i=1}^{k} a_i\mu([A_i]) \geq s$.

Hence, $M \models \sum_{i=1}^{k} a_i P^*(A_i) \geq s$. Using the similar explanation we can conclude the same for $a_k < 0$, having in mind that we must choose the rational numbers for which $r_k \leq s_k$.

Conversely, let us assume that $M \models \Phi$ and that assertion holds for all weighted formulas with less then k primitive weighted terms. For $\Phi = a_1 P^*(A_1) + a_2 P^*(A_2) + \cdots + a_k P^*(A_k) \geq s$, $a_k > 0$ and $\mu([A_i]) = s_i$, for $i = 1, k$, we have:

$M \models \sum_{i=1}^{k} a_i P^*(A_i) \geq s$, i.e., $\sum_{i=1}^{k} a_i\mu([A_i]) \geq s$,

$\sum_{i=1}^{k-1} a_i\mu([A_i]) \geq s - a_k\mu([A_i])$.

For every $t_k \in \mathbf{Q}$ and $t_k \geq s_k$, $\sum_{i=1}^{k-1} a_i\mu([A_i]) \geq s - a_k t_k$ holds. Hence,

$M \models \sum_{i=1}^{k-1} a_i P^*(A_i) \geq s - a_k t_k$

$\sum_{i=1}^{k-1} a_i P^*(A_i) \geq s - a_k t_k \in \overline{T}$, by the induction hypothesis.

By the induction hypothesis, we can also conclude that for every rational number $r_k \leq s_k$

$P^*(A_k) \geq r_k \in \overline{T}$.

Thus,

$\sum_{i=1}^{k} a_i P^*(A_i) \geq s - a_k(t_k - r_k) \in \overline{T}$, using axioms 2g and 2h.

Now, we can choose a sequences $\{r_k^n\}_n$, $\{t_k^n\}_n$ of rational numbers, such that $0 < s_k - r_k^n \leq \frac{1}{2n}$ and $0 < t_k^n - s_k \leq \frac{1}{2n}$, thus $0 < t_k^n - r_k^n \leq \frac{1}{n}$. All formulas of the form $\sum_{i=1}^{k} a_i P^*(A_i) \geq s - a_k(t_k^n - r_k^n)$ belong to $\overline{T}$. By the Theorem 5.5.7, we can conclude

$\sum_{i=1}^{k} a_i P^*(A_i) \geq s \in \overline{T}$.

For $a_k < 0$, the proof is analog to previous one. The only difference concerns the choice of the sequences $\{r_k^n\}_n, \{t_k^n\}_n$, where $r_k^n \geq s_k$ and $t_k^n \leq s_k$. □

6. CONCLUSION

In this paper we have presented a real-valued probability logic which allow Boolean combinations of formulas of the form $a_1 w(A_1) + \cdots + a_k w(A_k) \geq c$, where $A_1, \ldots, A_k$ are propositional formulas, and $a_1, \ldots, a_k, c$ are rational numbers. We have given an axiomatic system which contains an infinitary inference rule and have proved the corresponding soundness and extended completeness theorem. In [4] the similar results for finitely-valued probabilities (the corresponding logic was introduced in [6]) was presented. That proof was substantially different from our because we have considered real-valued probability functions.

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