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BOUNDARY DOMINATION IN GRAPHS

KM. Kathiresan[1], G. Marimuthu[2] and M. Sivanandha Saraswathy[3]

[1]*Center for Research and Post Graduate Studies in Mathematics*
Ayya Nadar Janaki Ammal College
Sivakasi - 626 124, Tamil Nadu, INDIA
(e-mail: kathir2esan@yahoo.com)

[2]*Department of Mathematics, The Madura College*
Madurai-625 011, Tamil Nadu, INDIA
(e-mail: yellowmuthu@yahoo.com)

[3]*Department of Mathematics*
Thiagarajar College of Engineering
Madurai - 625 006, Tamil Nadu, INDIA
(e-mail: sivanandha@tce.edu)

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Abstract. Let G be a nontrivial connected graph. The distance between two vertices u and v of G is the length of a shortest u-v path in G. Let u be a vertex in G. A vertex v is an eccentric vertex of u if $d(u, v) = e(u)$, that is every vertex at greatest distance from u is an eccentric vertex of u. A vertex v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. Consequently, if v is an eccentric vertex of u and w is a neighbor of v, then $d(u, w) \leq d(u, v)$. A vertex v may have this property, however, without being an eccentric vertex of u. A vertex v is a boundary vertex of a vertex u if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$. A vertex u may have more than one boundary vertex at different distance levels. A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. The number of boundary neighbors of a vertex u is called the boundary degree of u. In this paper, first we show that there is no relationship between the traditional degree and the boundary degree of a vertex. Finally we define boundary dominating set for a graph and we give an upper bound for the boundary domination number of a graph.

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1. INTRODUCTION AND DEFINITIONS

For graph-theoretical terminology and notations not defined here we follow Buckley [1], West [10] and Haynes *et al.,* [5]. Let G be a nontrivial connected graph. The distance between two vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. A vertex v is called an eccentric vertex of u if $e(u) = d(u, v)$. A vertex v is an eccentric vertex of G if v is an eccentric vertex of some vertex of G. Consequently if v is an eccentric vertex of u and w is a neighbor of v, then $d(u, w) \leq d(u, v)$. A vertex v may have this property, however, without being an eccentric vertex of u. A vertex v is a boundary vertex of u if $d(u, w) \leq d(u, v)$ for all $w \in N(v)$ (see [3, 4]). Chartrand *et al.,* [3] defined the boundary degree of a vertex as follows:

The boundary degree $b(v)$ of a vertex v in a connected graph G of order $n \geq 2$ is the number of vertices u in G having v as a boundary vertex.

In the classical sense,the degree of a vertex v is the number of vertices other than v at minimum distance from v. A vertex v in a graph is called a complete vertex if $\langle N(v) \rangle$ is complete. In the standard definition of domination in graphs, a vertex u dominates itself and each of its neighbors. One variation of domination is seen in [2] where the concept of detour domination in graphs is defined. The detour distance $D(u, v)$ from a vertex u to a vertex v is defined as the length of a longest u-v path in G. For a vertex v in G, define $D^{-}(v) = \ min\{D(u, v) : u \in V(G) - \{u\}\}$. A vertex $u(\neq v)$ is called a detour neighbor of v if $D(u, v) = D^{-}(v)$. A vertex v is said to detour dominate a vertex u if $u = v$ or u is a detour neighbor of v. A set S of vertices of G is called a detour dominating set if every vertex of G is detour dominated by some vertex in S. A detour dominating set of G of minimum cardinality is a minimum detour dominating set and this cardinality is the detour domination number of G.

If X and Y are two cities, then for a taxi driver the distance between the two cities is the actual distance between X and Y. However for a mobusal bus driver, the distance between the same cities is greater than the usual distance since he has to visit important places in and around the two cities to pickup and drop off passengers. So a mobusal bus driver has to find a shortest route that begins at X, ends at Y, and passes through each of the neighboring places of X and Y.

Karthiresan and Marimuthu [6, 9] discussed a variation of distance that models the bus route just described. For a simple connected graph G and for two vertices u and v of G, let $D_{u,v} = N(u) \cup N(v)$. We define a $D_{u,v}$- walk as a u-v walk in G that contains every vertex of $D_{u,v}$. The superior distance $d_D(u, v)$ from u to v is the length of a shortest $D_{u,v}$- walk. Another variation of domination is seen in [7, 9]. For each vertex $u \in V(G)$,define $d_D^-(u) = \ min\{d_D(u, v) : v \in V(G) - \{u\}\}$. A vertex $v(\neq u)$ is called a superior neighbor of u if $d_D(u, v) = d_D^-(u)$. A vertex u is said to superior dominate a vertex v if v is a superior neighbor of u. A set S of vertices of G is called a superior dominating set if every vertex of $V - S$ is superior dominated by some vertex of S. A superior dominating set of minimum cardinality is called minimum superior dominating set and its cardinality is the superior domination number of G.

A vertex v may have more than one boundary vertex at different distance levels. A vertex w is a boundary neighbor of a vertex u if w is a nearest boundary of u. Now we define $d^*(u) = \ min\{d(u, w) : w$ is a boundary vertex of $u\}$. A vertex w is a boundary neighbor of a vertex u if $d^*(u) = d(u, w)$. The set of all boundary neighbors of a vertex u is denoted by $N_b(u)$. The boundary degree $deg_b(u)$ of a vertex u is the number of boundary neighbors of u. We denote δ_b by the minimum among the boundary degree of vertices of G and Δ_b by the maximum among the boundary degree of vertices of G. A connected graph G is called an k-boundary regular if $\delta_b(G) = \Delta_b(G) = k$. A vertex u boundary dominate a vertex v if v is a boundary neighbor of u. A subset S of $V(G)$ is called a boundary dominating set if every vertex of $V - S$ is boundary dominated by some vertex of S. The minimum taken overall boundary dominating sets of a graph G is called the boundary domination number of G and is denoted by $\gamma_b(G)$.

Proposition 1. *In a complete graph G, $deg(v) = deg_b(v)$ for all $v \in V(G)$.*

The converse of the above proposition need not be true. For example, in any odd cycle C_{2m+1}, $m \geq 2$, $deg(v) = deg_b(v) = 2$ for all $v \in V(C_{2m+1})$.

Next we prove that there is no relationship between the traditional degree and the boundary degree of a vertex.

Theorem 1. *For each pair of positive integers a and b, there exists a connected graph G having a vertex v such that $deg(v) = a$ and $deg_b(v) = b$.*

Proof. Let a and b be two positive integers.

When $a = 1$, the graph $K_{1,b+1}$ has a vertex v such that $deg(v) = 1$ and $deg_b(v) = b$. Consider the cycle C_5 whose vertices are v_1, v_2, v_3, v_4 and v_5. The vertices v_3 and v_4 are the eccentric vertices of v_1. In C_5, $deg(v_1) = deg_b(v_1) = 2$.

Case 1. $a = b$, $a \geq 3$.

Consider the $a-2$ copies of P_3 with vertices $v_{1i}, v_{2i}, \ldots, v_{(a-2)i}, i = 1, 2, 3$ respectively. Merge the vertices v_{j1}, $j = 1, 2, \ldots, a-2$ with the vertex v_1 and let the resulting graph be H. Then $deg(v_1) = a = deg_b(v_1)$.

Case 2. $a < b$

Construct the graph H as in the above case. Now introduce k new vertices $u_1, u_2, \ldots, u_k$ such that $a + k = b$ and join these vertices with any one of the $v'_{j2}s$, $j = 1, 2, \ldots, a-2$. Let the resulting graph be H_1. Then $deg(v_1) = a$ and $deg_b(v_1) = a - 2 + 2 + b - a = b$ in H_1.

Case 3. $a > b$.

Construct the graph H as in the Case 1. Consider the $a-b$ copies of P_4 and let the end vertex of each P_4 be $v_{11}, v_{21}, \ldots, v_{a-b}$. Merge these end vertices with the vertex v_1 in H and let the resulting graph be H_2. Then deg $(v_1) = b-2+2+a-b = a$ and $deg_b(v_1) = b$ in H_2. □

Theorem 2. *For given positive integers a and b and a given graph G, there exists a connected graph having a vertex v such that $deg(v) = a$ and $deg_b(v) = b$ in which G is a subgraph.*

Proof. Let a and b be given positive integers and let G be the given graph. Let $u \in V(G)$. Introduce a new vertex v and join it with u. Let the resulting graph be H_1.

Case 1. $a = b$.

Let $C = N_b(v)$ and let $d(v, x) = l$ for any $x \in C$. Attach a path of length $k > 0$ at each vertex of C except at a vertex w (say) and attach $b-1$ copies of paths of length l with the vertex v in H_1. Let the resulting graph be H_2. Then $deg(v) = deg_b(v) = a$ in H_2.

Case 2. $a < b$.

Construct the graph H_2 as in Case 1. Let y be a vertex adjacent to w in H_2. Attach k paths of length 1 such that $a - 1 + 1 + k = b$ with the vertex y. Let the resulting graph be H_3. Then $deg(v) = a$ and $deg_b(v) = a - 1 + 1 + b - a = b$ in H_3.

Case 3. $a > b$.

Construct the graph H_2 as in Case 1. Attach $a - b$ copies of path of length $l + 1$ with the vertex v. Let the resulting graph be H_4. Then $deg(v) = a$ and $deg_b(v) = b$ in H_4. □

The graph $P_{2n}, n \geq 1$ are 1-boundary regular and the cycles $C_{2m+1}, m \geq 1$ are 2-boundary regular graphs. We expect that there are k-boundary regular graphs for every k.

Conjecture 1. *For each positive integer k, there exists a k-boundary regular graph H.*

Having a characterization theorem for a sequence to be a graphic sequence, it is natural to ask whether a sequence is boundary graphic sequence or not. Answering this question seems to be a difficult problem.

First we find the boundary domination number of some well known graphs and then an upper bound for γ_b.

Result 1. $\gamma_b(K_{m,n}) = 2$, $m, n \geq 2$.

Proof. Let V_1 and V_2 be bipartition of the vertex set of $K_{m,n}$. Let $u \in V_1$. Then $d(u, v) = 2$ for all $v \in V_1 - \{u\}$ and every vertex v in V_1 is a boundary neighbour of u except u. Similarly if $u \in V_2$, then every vertex of V_2 is a boundary neighbour of u except u. Thus $\gamma_b(K_{m,n}) = 2$. □

Proposition 2. *Let T be a tree of order n with n_1 pendent vertices. Then $\gamma_b(T) = n - n_1$.*

Proof. Let V_1 be the set of all pendant vertices of the tree T of order n_1. Then every vertex in $V - V_1$ has a boundary neighbour in V_1. Then $\gamma_b(T) = n - n_1$. □

Corollary 1. *For any path P_n, $n \geq 3$, $\gamma_b(P_n) = n - 2$.*

Proposition 3. *For a complete graph $K_n, \gamma_b(K_n) = 1$.*

The converse of the above proposition need not be true. For example, $\gamma_b(K_{1,n}) = 1$, $n \geq 1$.

Theorem 3. *For a given $m \geq 2$, there exists a connected graph G such that $\gamma(G) = 2$ and $\gamma_b(G) = m$.*

Proof. For a given $m \geq 2$, choose a path P_n such that $n = 3m - 2$ whose vertices are $v_1, v_2, \ldots, v_n$. When $m = 2, \gamma(P_n) = 2$, by Corollary 2.3. Now assume that $m \geq 3$. Consider $\overline{P_n}$. If $r(P_n) \geq 2$ then $\overline{P_n}$ is a self centered graph of radius 2. Then the set $S = \{v_{3i+2} : i = 0, 1, 2, \ldots, \frac{n-4}{3}\} \cup \{v_n\}$ is a boundary dominating set with minimum cardinality. Thus $\gamma_b(\overline{P_n}) = \frac{n-4}{3} + 1 + 1 = m$. It is clear that $\gamma(\overline{P_n}) = 2$. Let $G = \overline{P_n}$ and hence G is the required graph. □

Our next result establishes an upper bound on the boundary domination number of connected graphs. First, we make a proposition.

Proposition 4. *Let u be a vertex of a connected graph G. Then $V - N_b(u)$ is a boundary dominating set for G.*

Theorem 4. *If G is a connected graph of order $n \geq 3$, then $\gamma_b(G) \leq n - 2$.*

Proof. Let u be a vertex of a connected graph G. Then by the above proposition, $V - N_b(u)$ is a boundary dominating set. But $|N_b(u)| \geq 1$. Thus $\gamma_b(G) \leq n - 1$. Suppose $\gamma_b(G) = n - 1$. Then there exists a unique vertex u^* in G such that u^* is a boundary neighbour of every vertex of $V - \{u^*\}$, this is a contradiction to the fact that in a graph there exist at least two boundary vertices. Thus $\gamma_b(G) \leq n - 2$. □

By Corollary 2.3, it is clear that the sharpness of the upper bound is satisfied by the paths P_n, $n \geq 3$.

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