Kragujevac J. Math. 33 (2010) 95–99.

ON A CONVERSE OF KY FAN INEQUALITY

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(Received December 14, 2009)

Abstract. In this short note a converse and an improvement of the famous Ky Fan inequality is given.

2010 Mathematics Subject Classification: 26D15.
Key words: Ky Fan inequality, Global bounds, Converses.

1. INTRODUCTION

Throughout the paper we denote by $\mathbf{x} = \{x_i\}$ a finite sequence of positive numbers and with $\mathbf{w} = \{w_i\}, \sum w_i = 1$ a sequence of positive weights associated with $\mathbf{x}$.

Denote also by

$$A(\mathbf{w}, \mathbf{x}) := \sum w_i x_i; G(\mathbf{w}, \mathbf{x}) := \prod x_i^{w_i},$$

the generalized arithmetic and geometric means of numbers x_i, respectively.

The well-known arithmetic-geometric inequality states that, for arbitrary $\mathbf{x}, \mathbf{w}$,

$$A(\mathbf{w}, \mathbf{x}) = \sum w_i x_i \geq \prod x_i^{w_i} = G(\mathbf{w}, \mathbf{x}). \tag{1}$$

The most celebrated counterpart of A-G inequality is the inequality of Ky Fan which says that

$$\frac{\sum w_i x_i}{\sum w_i(1-x_i)} \geq \frac{\prod x_i^{w_i}}{\prod (1-x_i)^{w_i}} \tag{2}$$

whenever $x_i \in (0, 1/2]$.

Since its publication in 1961 [2, p.5], Fan's result has evoked a great interest and a plenty of different proofs as well as noteworthy extensions and refinements are given (cf. [1], [3]).

Although it is not at all obvious, the Ky Fan inequality is stronger than A-G inequality. Indeed, for an arbitrary finite sequence $\{x_i\}$ of positive numbers, consider the sequence $\{x_i/t\}$ with $t \geq 2 \max x_i$. Since $x_i/t \in (0, 1/2]$, applying (2) we get

$$\frac{\sum w_i x_i}{\sum w_i(1-x_i/t)} \geq \frac{\prod x_i^{w_i}}{\prod (1-x_i/t)^{w_i}}.$$

Letting $t \to \infty$ we obtain (1). Therefore, the A-G inequality is a consequence of Ky Fan inequality.

The aim of this paper is to give a global converse of the inequality (2), that is, a converse which does not depend on $\mathbf{w}$ or $\mathbf{x}$ but only on an interval I where x_i belong.

An upper global bound for Jensen's inequality was given by Dragomir in [4].

Theorem A *If f is a differentiable convex mapping on $I := [a, b]$, then we have*

$$\sum w_i f(x_i) - f(\sum w_i x_i) \leq \frac{1}{4}(b-a)(f'(b) - f'(a)) := T_f(a, b).$$

In [5] we obtain an upper global bound without differentiability restriction on f.

Theorem B *For any $\mathbf{w}$ and $\mathbf{x} \in [a, b]$, we have*

$$(0 \leq) \sum w_i f(x_i) - f(\sum w_i x_i) \leq f(a) + f(b) - 2f(\frac{a+b}{2}) := S_f(a, b)$$

for any f that is convex over $I := [a, b]$.

Although the bounds T and S are not comparable in general, for a plenty of cases the bound $S_f(a, b)$ is better than $T_f(a, b)$. For instance, for a convex function f given by

$$f(x) = -x^s, s \in (0,1); \quad f(x) = x^s, s \in (-\infty, 0) \cup (1, +\infty); \; I \subset R^+,$$

we have that

$$S_f(a,b) \le T_f(a,b),$$

for each $s \in (-\infty, 0) \cup (0,1) \cup (1,2] \cup [3, +\infty)$.

2. RESULTS

The form of a converse of Ky Fan inequality is given in the sequel.

Theorem 1. *If* $0 < a \le x_i \le b \le 1/2$, *then*

$$\frac{G(\mathbf{w},\mathbf{x})}{G(\mathbf{w},\mathbf{1}-\mathbf{x})} \le \frac{A(\mathbf{w},\mathbf{x})}{A(\mathbf{w},\mathbf{1}-\mathbf{x})} \le \frac{G(\mathbf{w},\mathbf{x})}{G(\mathbf{w},\mathbf{1}-\mathbf{x})} S(a,b), \tag{3}$$

where $S(a,b) = \dfrac{(1-a)(1-b)(a+b)^2}{ab(2-a-b)^2}$.

Applying Theorem A another converse could be established, that is,

$$\frac{G(\mathbf{w},\mathbf{x})}{G(\mathbf{w},\mathbf{1}-\mathbf{x})} \le \frac{A(\mathbf{w},\mathbf{x})}{A(\mathbf{w},\mathbf{1}-\mathbf{x})} \le \frac{G(\mathbf{w},\mathbf{x})}{G(\mathbf{w},\mathbf{1}-\mathbf{x})} T(a,b),$$

with $T(a,b) = \exp\left[\dfrac{(b-a)^2(1-a-b)}{4ab(1-a)(1-b)}\right]$.

But we shall show in the sequel that for each $[a,b] \subset (0,1/2]$ the estimation $S(a,b)$ is better than $T(a,b)$.

In the case of uniform weights we give the following improvement of Ky Fan inequality (note that $S(a,b) > 1$ unless $a = b$).

Theorem 2. *Let* $\mu := \min_{1 \le i \le n} x_i$, $\nu := \max_{1 \le i \le n} x_i$; $0 < \mu \le \nu \le 1/2$. *Then*

$$\frac{\sum_1^n x_i}{\sum_1^n (1-x_i)} \ge \left[\frac{\prod_1^n x_i}{\prod_1^n (1-x_i)} S(\mu,\nu)\right]^{1/n}, \tag{4}$$

and this estimation is sharp.

Combining this assertion with the previous one, we obtain the following refinement of Ky Fan inequality in the case of uniform weights.

Theorem 3. *Let $\mu, \nu, S(\cdot,\cdot)$ be defined as above. We have*

$$\left[\frac{\prod_1^n x_i}{\prod_1^n (1-x_i)} S(\mu,\nu)\right]^{1/n} \leq \frac{\sum_1^n x_i}{\sum_1^n (1-x_i)} \leq \left[\frac{\prod_1^n x_i}{\prod_1^n (1-x_i)}\right]^{1/n} S(\mu,\nu). \tag{5}$$

3. PROOFS

Proof of Theorem 1

Considering the function $f(x) = \log \dfrac{1-x}{x}$, $f''(x) = \dfrac{1-2x}{x^2(1-x)^2}$, we conclude that it is convex on $(0, 1/2]$. Therefore, applying Theorem B in this case, we get

$$0 \leq \sum w_i \log \frac{1-x_i}{x_i} - \log \frac{1-\sum w_i x_i}{\sum w_i x_i} \leq \log \frac{1-a}{a} + \log \frac{1-b}{b} - 2\log \frac{1-\frac{a+b}{2}}{\frac{a+b}{2}},$$

that is,

$$0 \leq \log \frac{\sum w_i x_i}{\sum w_i (1-x_i)} - \log \frac{\prod x_i^{w_i}}{\prod (1-x_i)^{w_i}} \leq \log S(a,b),$$

and the result follows. □

Proof that $S(a,b) \leq T(a,b)$

Applying elementary inequalities $1+t \leq \exp t$, $(u+v)^2 \geq 4uv$, we get

$$\begin{aligned} S(a,b) &= 1 + \frac{(1-a-b)(b-a)^2}{ab(2-a-b)^2} \leq \exp\left[\frac{(1-a-b)(b-a)^2}{ab(2-a-b)^2}\right] \\ &\leq \exp\left[\frac{(1-a-b)(b-a)^2}{4ab(1-a)(1-b)}\right] = T(a,b). \end{aligned}$$

□

Proof of Theorem 2

This is a direct consequence of [5 , Theorem C] which says

If $\mu := \min_{1\leq i\leq n} x_i, \nu := \max_{1\leq i\leq n} x_i$ *and* f *is convex on the interval* $[\mu,\nu]$, *then*

$$\frac{1}{n}\left(\sum_1^n f(x_i)\right) - f\left(\frac{1}{n}(\sum_1^n x_i)\right) \geq \frac{1}{n}\left(f(\mu) + f(\nu) - 2f(\frac{\mu+\nu}{2})\right),$$

applied on the function $f(x) = \log \frac{1-x}{x}$. □

References

[1] Alzer, H., *Refinements of Ky Fan's inequality*, Proc. Amer. Math. Soc. **117/2** (1993), 159-165.

[2] Beckenbach, E. F, and Bellman, R., *Inequalities*, Springer, Berlin, 1961.

[3] Bullen, P. S., Mitrinovic, D. S. and Vasic, P. M., *Means and their inequalities*, Reidel, Dordrecht, 1988.

[4] Dragomir, S. S., *A converse result for Jensen's discrete inequality via Grüss inequality and applications in Information Theory*, Analele Univ. Oradea. Fasc. Math., **7** (1999-2000), 178-189.

[5] Simic, S., *Jensen's inequality and new entropy bounds*, Appl. Math. Lett. **22** (2009), 1262-1265.