ON A CONVERSE OF KY FAN INEQUALITY

Slavko Simić

Mathematical Institute SANU,
Kneza Mihaila 36, 11000 Belgrade, Serbia
(e-mail: ssimic@turing.mi.sanu.ac.rs)

(Received December 14, 2009)

Abstract. In this short note a converse and an improvement of the famous Ky Fan inequality is given.

2010 Mathematics Subject Classification: 26D15.

Key words: Ky Fan inequality, Global bounds, Converses.

1. INTRODUCTION

Throughout the paper we denote by \( x = \{x_i\} \) a finite sequence of positive numbers and with \( w = \{w_i\}, \sum w_i = 1 \) a sequence of positive weights associated with \( x \).

Denote also by

\[
A(w, x) := \sum w_i x_i; \quad G(w, x) := \prod x_i^{w_i},
\]

the generalized arithmetic and geometric means of numbers \( x_i \), respectively.

The well-known arithmetic-geometric inequality states that, for arbitrary \( x, w \),

\[
A(w, x) = \sum w_i x_i \geq \prod x_i^{w_i} = G(w, x).
\]
The most celebrated counterpart of A-G inequality is the inequality of Ky Fan which says that
\[
\sum w_i x_i \sum w_i (1 - x_i) \geq \prod x_i^{w_i} \prod (1 - x_i)^{w_i}
\]
whenever \( x_i \in (0, 1/2] \).

Since its publication in 1961 [2, p.5], Fan’s result has evoked a great interest and a plenty of different proofs as well as noteworthy extensions and refinements are given (cf. [1], [3]).

Although it is not at all obvious, the Ky Fan inequality is stronger than A-G inequality. Indeed, for an arbitrary finite sequence \( \{x_i\} \) of positive numbers, consider the sequence \( \{x_i/t\} \) with \( t \geq 2 \max x_i \). Since \( x_i/t \in (0, 1/2] \), applying (2) we get
\[
\sum w_i x_i \sum w_i (1 - x_i/t) \geq \prod x_i^{w_i} \prod (1 - x_i/t)^{w_i}.
\]
Letting \( t \to \infty \) we obtain (1). Therefore, the A-G inequality is a consequence of Ky Fan inequality.

The aim of this paper is to give a global converse of the inequality (2), that is, a converse which does not depend on \( w \) or \( x \) but only on an interval \( I \) where \( x_i \) belong.

An upper global bound for Jensen’s inequality was given by Dragomir in [4].

**Theorem A** If \( f \) is a differentiable convex mapping on \( I := [a, b] \), then we have
\[
\sum w_i f(x_i) - f(\sum w_i x_i) \leq \frac{1}{4} (b - a)(f'(b) - f'(a)) := T_f(a, b).
\]

In [5] we obtain an upper global bound without differentiability restriction on \( f \).

**Theorem B** For any \( w \) and \( x \in [a, b] \), we have
\[
(0 \leq) \sum w_i f(x_i) - f(\sum w_i x_i) \leq f(a) + f(b) - 2f\left(\frac{a + b}{2}\right) := S_f(a, b)
\]
for any \( f \) that is convex over \( I := [a, b] \).

Although the bounds \( T \) and \( S \) are not comparable in general, for a plenty of cases the bound \( S_f(a, b) \) is better than \( T_f(a, b) \). For instance, for a convex function \( f \) given
by
\[ f(x) = -x^s, \; s \in (0, 1); \quad f(x) = x^s, \; s \in (-\infty, 0) \cup (1, +\infty); \quad I \subset \mathbb{R}^+, \]
we have that
\[ S_f(a, b) \leq T_f(a, b), \]
for each \( s \in (-\infty, 0) \cup (0, 1) \cup (1, 2] \cup [3, +\infty) \).

2. RESULTS

The form of a converse of Ky Fan inequality is given in the sequel.

**Theorem 1.** If \( 0 < a \leq x_i \leq b \leq 1/2 \), then
\[
\frac{G(w, x)}{G(w, 1-x)} \leq \frac{A(w, x)}{A(w, 1-x)} \leq \frac{G(w, x)}{G(w, 1-x)} S(a, b),
\]
where \( S(a, b) = \frac{(1-a)(1-b)(a+b)^2}{ab(2-a-b)^2} \).

Applying Theorem A another converse could be established, that is,
\[
\frac{G(w, x)}{G(w, 1-x)} \leq \frac{A(w, x)}{A(w, 1-x)} \leq \frac{G(w, x)}{G(w, 1-x)} T(a, b),
\]
with \( T(a, b) = \exp \left[ \frac{(b-a)^2(1-a-b)}{4ab(1-a)(1-b)} \right] \).

But we shall show in the sequel that for each \([a, b] \subset (0, 1/2] \) the estimation \( S(a, b) \) is better than \( T(a, b) \).

In the case of uniform weights we give the following improvement of Ky Fan inequality (note that \( S(a, b) > 1 \) unless \( a = b \)).

**Theorem 2.** Let \( \mu := \min_{1 \leq i \leq n} x_i, \nu := \max_{1 \leq i \leq n} x_i; \; 0 < \mu \leq \nu \leq 1/2 \). Then
\[
\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n (1-x_i)} \geq \left[ \frac{\prod_{i=1}^n x_i}{\prod_{i=1}^n (1-x_i)} S(\mu, \nu) \right]^{1/n},
\]
and this estimation is sharp.
Combining this assertion with the previous one, we obtain the following refinement of Ky Fan inequality in the case of uniform weights.

**Theorem 3.** Let \( \mu, \nu, S(\cdot, \cdot) \) be defined as above. We have

\[
\left[ \frac{\prod_i^n x_i}{\prod_i^n (1 - x_i)} S(\mu, \nu) \right]^{1/n} \leq \frac{\sum_i^n x_i}{\sum_i^n (1 - x_i)} \leq \left[ \frac{\prod_i^n x_i}{\prod_i^n (1 - x_i)} \right]^{1/n} S(\mu, \nu). \tag{5}
\]

3. PROOFS

**Proof of Theorem 1**

Considering the function \( f(x) = \log \frac{1 - x}{x}, f''(x) = \frac{1 - 2x}{x^2(1 - x)^2} \), we conclude that it is convex on \((0, 1/2]\). Therefore, applying Theorem B in this case, we get

\[
0 \leq \sum w_i \log \frac{1 - x_i}{x_i} - \log \frac{\sum w_i x_i}{\sum w_i (1 - x_i)} \leq \log \frac{1 - a}{a} + \log \frac{1 - b}{b} - 2 \log \frac{1 - \frac{a + b}{2}}{\frac{a + b}{2}},
\]

that is,

\[
0 \leq \log \frac{\sum w_i x_i}{\sum w_i (1 - x_i)} - \log \frac{\prod_i^n x_i^{w_i}}{\prod_i^n (1 - x_i)^{w_i}} \leq \log S(a, b),
\]

and the result follows.

**Proof that** \( S(a, b) \leq T(a, b) \)

Applying elementary inequalities \( 1 + t \leq \exp t, (u + v)^2 \geq 4uv \), we get

\[
S(a, b) = 1 + \frac{(1 - a - b)(b - a)^2}{ab(2 - a - b)^2} \leq \exp \left[ \frac{(1 - a - b)(b - a)^2}{ab(2 - a - b)^2} \right] \leq \exp \left[ \frac{(1 - a - b)(b - a)^2}{4ab(1 - a)(1 - b)} \right] = T(a, b).
\]

**Proof of Theorem 2**

This is a direct consequence of [5, Theorem C] which says

*If \( \mu := \min_{1 \leq i \leq n} x_i, \nu := \max_{1 \leq i \leq n} x_i \) and \( f \) is convex on the interval \([\mu, \nu]\), then*

\[
\frac{1}{n} \left( \sum_{i=1}^n f(x_i) \right) - f\left( \frac{1}{n} \left( \sum_{i=1}^n x_i \right) \right) \geq \frac{1}{n} \left( f(\mu) + f(\nu) - 2f\left( \frac{\mu + \nu}{2} \right) \right),
\]

*applied on the function \( f(x) = \log \frac{1 - x}{x} \).*
References


