A GENERALIZATION OF QI’S INEQUALITY FOR SUMS

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Abstract. By a majorization method, a pair of inequalities for sums of nonnegative se-
quences are established, and so an open problem posed by F. Qi is resolved.

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1. INTRODUCTION

In [4], the following inequality between the sum of squares and the exponential of
sum of a nonnegative sequence was obtained: For \((x_1, x_2, \ldots, x_n) \in R^n_+\) and \(n \geq 2\),
the inequality
\[
\frac{e^2}{4} \sum_{i=1}^{n} x_i^2 \leq \exp \left( \sum_{i=1}^{n} x_i \right)
\]
(1)
is valid, where \(R^n_+ = \{(x_1, \ldots, x_n) \in R^n : x_i \geq 0, i = 1, \ldots, n\}\). The equality in (1)
holds if \(x_i = 2\) and \(x_j = 0\) for some given \(1 \leq i \leq n\) and all \(1 \leq j \leq n\) with \(j \neq i\).
The constant \(\frac{e^2}{4}\) in the inequality(1) is the best possible.
The first open problem in [4] may be quoted as follows: For \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+\) and \(n \geq 2\), determine the best possible constants \(\alpha_n, \lambda_n \in \mathbb{R}\) and \(0 < \beta_n, \mu_n < \infty\) such that
\[
\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leq \exp \left( \sum_{i=1}^{n} x_i \right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{2}
\]

First of all, we claim that the right-hand side inequality in (2) is generally untenable. In fact, when \(n = 2\), the right-hand side inequality in (2) becomes
\[
e^{x_1 + x_2} \leq \mu_2 \left( x_1^{\lambda_2} + x_2^{\lambda_2} \right). \tag{3}
\]
Further taking \(x_2 = 0\) in the inequality (3) reduces
\[
e^{x_1} \leq \mu. \tag{4}
\]
For any given \(\lambda > 0\), the function \(\frac{e^{x_1}}{x_1^{\lambda}}\) tends to \(\infty\) as \(x_1 \to \infty\). Hence, the inequality (4) does not hold if \(x_1\) is large enough.

In this short note, by using a method in the theory of majorization, we give an affirmative solution to the left-hand side inequality in (2), which is also a generalization of the inequality (1), as follows.

**Theorem 1.** Let \((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+\) and \(n \geq 2\). If \(\alpha \geq 1\), then the inequality
\[
\frac{e^{\alpha}}{\alpha^{\alpha}} \left( \sum_{i=1}^{n} x_i^{\alpha} \right) \leq \exp \left( \sum_{i=1}^{n} x_i \right) \tag{5}
\]
is valid. The equality in (5) holds if and only if \(x_i = \alpha\) and \(x_j = 0\) for some given \(1 \leq i \leq n\) and all \(1 \leq j \leq n\) with \(j \neq i\).

**Theorem 2.** Let \(\{x_i\}_{i=1}^{\infty}\) be a nonnegative sequence such that \(\sum_{i=1}^{\infty} x_i < \infty\). For \(\alpha \geq 1\), the inequality
\[
\frac{e^{\alpha}}{\alpha^{\alpha}} \sum_{i=1}^{\infty} x_i^{\alpha} \leq \exp \left( \sum_{i=1}^{\infty} x_i \right) \tag{6}
\]
is valid.
2. DEFINITIONS AND LEMMAS

In order to prove our theorems, the following definitions and lemmas are needed.

**Definition 1.** [1, 6] Let \( x = (x_1, \ldots, x_n) \in R^n \) and \( y = (y_1, \ldots, y_n) \in R^n \).

1. The sequence \( x \) is said to be majorized by \( y \) (in symbols \( x \preceq y \)) if \( \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \) for \( k = 1, 2, \ldots, n-1 \) and \( \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \), where \( x_1 \geq \cdots \geq x_n \) and \( y_1 \geq \cdots \geq y_n \) are rearrangements of \( x \) and \( y \) in a descending order, and \( x \) is said to strictly majorized by \( y \) (in symbols \( x \prec y \)) if \( x \) is not a permutation of \( y \).

2. A function \( f : \Omega \to R \) is said to be a strictly Schur-convex on \( \Omega \subset R^n \) if the relation \( x \preceq y \) on \( \Omega \) implies \( f(x) < f(y) \). A function \( f \) is said to be strictly Schur-concave on \( \Omega \) if and only if \(-f\) is strictly Schur-convex on \( \Omega \).

**Definition 2.** [6] Let set \( \Omega \subseteq R^n \). \( \Omega \) is said to be a convex set if \( x, y \in \Omega, 0 \leq \alpha \leq 1 \) implies \( \alpha x + (1 - \alpha) y = (\alpha x_1 + (1 - \alpha) y_1, \ldots, \alpha x_n + (1 - \alpha) y_n) \in \Omega \).

**Lemma 1.** [6, p. 5] Let \( \Omega \subset R^n \) be symmetric and have a nonempty interior convex set \( \Omega^0 \), and let \( f : \Omega \to R \) be continuous on \( \Omega \) and differentiable on \( \Omega^0 \). Then the function \( f \) is strictly Schur-convex (or Schur-concave respectively) on \( \Omega \) if and only if \( f \) is symmetric on \( \Omega \) and satisfies

\[
(x_1 - x_2) \left( \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) > 0 \quad (\text{or} < 0, \text{ respectively})
\]

for \( x = (x_1, x_2) \in \Omega^0 \) with \( x_1 \neq x_2 \).

**Lemma 2.** For any given positive real number \( s \) and \( \alpha \), we have

\[
\frac{e^\alpha}{\alpha^\alpha} \leq \frac{e^s}{s^\alpha}.
\]

The equality in (8) holds if and only if \( s = \alpha \).

**Proof.** Let \( \varphi(s) = \alpha \ln s - s \). Then \( \varphi'(s) = \frac{\alpha}{s} - 1 \leq 0 \) for \( s \geq \alpha > 0 \), which means that \( \varphi(s) \) is increasing, and \( \varphi'(s) \geq 0 \) for \( 0 < s \leq \alpha \), which means that \( \varphi(s) \) is
decreasing. Hence, for any $s > 0$, we have

$$
\varphi(s) = \alpha \ln s - s \leq \varphi(\alpha) = \alpha \ln \alpha - \alpha,
$$

i.e., the inequality (8) is valid and the equality in (8) holds if and only if $s = \alpha$. □

3. PROOFS OF THEOREMS

Now we are in a position to prove our theorems.

**Proof of Theorem 1**

Let

$$
f(x) = f(x_1, \ldots, x_n) = \ln \left( \sum_{i=1}^{n} x_i^\alpha \right) - s,
$$

where $s = \sum_{i=1}^{n} x_i$. Simple calculation gives

$$
\Delta := (x_1 - x_2) \left( \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} \right) = \frac{\alpha (x_1 - x_2)(x_1^{\alpha - 1} - x_2^{\alpha - 1})}{\sum_{i=1}^{n} x_i^\alpha}.
$$

When $\alpha > 1$, since $x^{\alpha - 1}$ is strictly increasing on $(0, \infty)$, it follows easily that $(x_1 - x_2)(x_1^{\alpha - 1} - x_2^{\alpha - 1}) > 0$ for $x_1 \neq x_2$, and then $\Delta > 0$, so, by Lemma 1, $f(x)$ is strictly Schur-convex on $R_+^n$. It is easy to see that

$$
x = (x_1, \ldots, x_n) \preceq \left( \underbrace{s, 0, \ldots, 0}_{n-1} \right) = y
$$

and $x \prec y$ unless $x_i = s$ and $x_j = 0$ for some given $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $j \neq i$. Hence,

$$
f(x_1, \ldots, x_n) = \ln \left( \sum_{i=1}^{n} x_i^\alpha \right) - s \leq f \left( s, \underbrace{0, \ldots, 0}_{n-1} \right) = \alpha \ln s - s,
$$

that is,

$$
\frac{e^s}{s^\alpha} \left( \sum_{i=1}^{n} x_i^\alpha \right) \leq \exp \left( \sum_{i=1}^{n} x_i \right),
$$

and the equality in (12) holds if and only if $x_i = s$ and $x_j = 0$ for some given $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $j \neq i$. Combining the inequality (12) with the inequality
(8) yields that the inequality (5) is valid and the equality in (5) holds if and only if \(x_i = \alpha\) and \(x_j = 0\) for some given \(1 \leq i \leq n\) and all \(1 \leq j \leq n\) with \(j \neq i\).

The proof of Theorem 1 is complete.

Proof of Theorem 2

Letting \(n \to \infty\) in Theorem 1 yields Theorem 2 readily.

Remark. After the preprint [5] of this paper was announced, there have been several papers such as [2, 3] dedicated to discuss the open problems posed by F. Qi in [4].

References


