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SOME GEOMETRICAL PROPERTIES OF MARGINALLY TRAPPED SURFACES IN MINKOWSKI SPACE M^4

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Abstract. Recently, many papers appeared about marginally trapped surfaces in various spaces [1], [2], [3], [4]. We present here for marginally trapped surfaces in Minkowski space some geometrical results related to the classical differential geometry of Darboux, Bianchi, Ribaucour, Guichard etc.

1. Introduction

The concept of trapped surfaces was introduced by Roger Penrose for its importance in general relativity and in the theory of cosmic black holes. A spatial surface in Minkowski space M^4 is said to be marginally trapped if its mean curvature vector H is lightlike at each point. Surfaces with lightlike mean curvature vector have been introduced by Radu Rosca in 1972 under the name of pseudo–minimal surface ([7], [8], [9]), for instance or quasi-minimal surfaces in [15].

2. Some geometrical properties of spacelike surfaces in M^4

Let M be a 2–dimensional surface in M^4. At each $m \in M$ we choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that the vectors e_1, e_2 span the tangent plane $T_m M$, and e_3, e_4 the normal plane $T_m^{\perp} M$ with $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \langle e_3, e_3\rangle = 1 = -\langle e_4, e_4\rangle$.

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Then we have

$$\begin{aligned} dm =& \omega^1 e_1 + \omega^2 e_2, \\ de_i =& \omega_i^j e_j, \\ \omega_i^j =& h_{ik}^j \omega^k. \end{aligned}$$

For each $\xi \in T_mM$ the shape operator A_ξ is a symmetric endomorphism of the tangent space T_mM at $m \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle,$$

where X, Y are vector fields tangent to M.

The mean curvature vector is $H = (h_{11}^3 + h_{22}^3)e_3 + (h_{11}^4 + h_{22}^4)e_4$.

We can introduce also the Kommerell conic K in $T_m^\perp M$. K is the locus of the points $N = Xe_3 + Ye_4$ in which $T_m^\perp M$ is cut by the neighboring normal planes of M.

The equation of K is

$$(\omega^1 - X\omega_1^3 + Y\omega_1^4) \wedge (\omega^2 - X\omega_2^3 + Y\omega_2^4) = 0,$$

or

$$\begin{aligned} &X^2[h_{11}^3 h_{22}^3 - (h_{12}^3)^2] + XY[-h_{11}^3 h_{22}^4 - h_{22}^3 h_{11}^4 + 2h_{12}^3 h_{12}^4] \\ &+Y^2[h_{11}^4 h_{22}^4 - (h_{12}^4)^2] - X[h_{11}^3 + h_{22}^3] + Y[h_{11}^4 + h_{22}^4] + 1 = 0. \end{aligned}$$

K degenerates into straight lines if $K_n = 0$.

2.1. FOCAL PROPERTIES OF THE ISOTROPIC STRAIGHT LINES OF $T_m^\perp M$ THROUGH m

We denote by D_+ and D_- the isotropic straight lines of the normal plane through m with direction e_3+e_4 and e_3-e_4. The family D_+ (respectively D_-) can be decomposed into 2 families of developable surfaces. The point $f = m + p(e_3 + e_4)$ can be chosen on D_+ such that the curve (f) describes an isotropic curve (with null tangent vector $e_3 + e_4$). We find then on each D_+ two points corresponding to the equation

$$\begin{aligned} &p^2[h_{11}^3 h_{22}^3 + h_{11}^4 h_{22}^4 - h_{22}^3 h_{11}^4 - h_{22}^4 h_{11}^3 - (h_{12}^3)^2 - (h_{12}^4)^2 - 2h_{12}^3 h_{12}^4] \\ &+p[-h_{11}^3 - h_{22}^3 + h_{11}^4 + h_{22}^4] + 1 = 0. \end{aligned} \tag{2.1}$$

It is possible to show that the curves corresponding on M to these developable define an orthogonal net. These kinds of families of straight lines were studied by Guichard [6] under the name of congruence I. We have then in the normal plane two lines D_+ and D_- belonging to two congruences I and on M two orthogonal nets N_+ and N_-.

If $N_+ = N_-$ it is possible to show that $K_n = 0$ and the common net is an orthogonal conjugate net (in Dupin sense) or O net [6].

2.2. CONNEXION WITH SPHERES CONGRUENCES IN EUCLIDEAN SPACE E^3

We consider an arbitrary Euclidean hyperplane E^3 of M^4 and a spatial surface M of M^4. The two isotropic normals D_+, D_- to M cut E^3 in two points m_+, m_-; using classical results (Vincensini [13]) we can show that the orthogonal projections of $D+$, D_- on E^3 are normal to the surfaces (m_+), (m_-). If ω is the projection of m on E^3, we see that the isotropic hypercone with center m cuts E^3 in a sphere Ω with center ω, containing m_+ and m_-, this sphere is tangent to (m_+) and (m_-) which are the focal surfaces of the sphere congruence (Ω). Any sphere congruence of E^3 can be associated to a spatial surface of M^4.

3. Some geometrical properties of marginally trapped surfaces

For a marginally trapped surface, the mean curvature vector is lightlike: $||H|| = 0$; then

$$h^3_{11} + h^3_{22} = \epsilon(h^4_{11} + h^4_{22}), \quad \epsilon = \pm 1.$$

For $\epsilon = 1$, H is parallel to $e_3 + e_4$ and then the equation (2.1) has two opposite solutions: m is the middle of the focal segment on D_+. The line D_+ cuts the conic K into two points symmetric w.r.t. m. Then the projection of m on E^3 is the middle of the focal segment on the normals to (m_+), and we have then the following result:

The projection of a marginally trapped surface M of M^4 on an arbitrary Euclidean hyperplane of M^4 is the middle surface of a normal congruence of straight lines associated with M ([10], [11], [12]).

It is easy to show that *to any middle surface of a normal congruence of E_3 we can associate a marginally trapped surface of M^4.* Then the study of marginally trapped surfaces of M^4 can be associated to the study of middle surfaces of normal congruence. Many papers are devoted to this subject from the end of 19th century to the beginning of the 20th. Many references can be found in works of Eisenhart [5], Vincensini [13], [14], Darboux, Ribaucour, Rosca, M. T. Calapso, Vaulot, ...

3.1. A PARTICULAR CASE

An interesting situation is the case of marginally trapped surfaces with null normal curvature $K_n = 0$. Then the developpables of the the congruences (D_+) and (D_-) on m are in correspondence and also the lines of curvature on the two sheets of the envelope of the congruence of spheres Ω. We obtain then a sphere congruence of Ribaucour. The congruence (D_+) has a projection which is conjugate to the conjugate net on Ω. We obtain then on E^3 a congruence of Ribaucour (congruences of Ribaucour are characterized by the following property: the developpables of a congruence of Ribaucour cut the middle surface in a conjugate system).

3.2. AN EXAMPLE OF CONSTRUCTION OF A MARGINALLY TRAPPED SURFACE

We use a classical torus in E^3 with equation in a frame ϵ_1, ϵ_2, ϵ_3,

$$x_1 = (R + r\cos\phi)\cos\theta, \quad x_2 = (R + r\cos\phi)\sin\theta, \quad x_3 = r\sin\phi.$$

The middle surface of the congruence of normals is

$$x_1 = \frac{R}{2}\cos\theta, \quad x_2 = \frac{R}{2}\sin\theta, \quad x_3 = -\frac{R}{2}\tan\phi.$$

It is a circular cylinder with axis $x_1 = 0$, $x_2 = 0$ and radius $R/2$.

We compute the orthonormal frame $\epsilon_1, \epsilon_2, \epsilon_3$ by ϵ_4 with $\langle \epsilon_4, \epsilon_4 \rangle = -1$ and obtain the marginally trapped surface associated with the torus. We can construct on it a moving frame $\{e_1, e_2, e_3, e_4\}$ where $e_1 = (\sin\theta, -\cos\theta, 0, 0)$, $e_2 = (0, 0, -1/\cos\phi, \tan\phi)$, $e_3 = (\cos\theta, \sin\theta, 0, 0)$, $e_4 = (0, 0, \tan\phi, -1/\cos\phi)$ and

$$\omega^1 = -\frac{R}{2}d\theta, \quad \omega^2 = \frac{R}{2}\frac{d\phi}{\cos\phi}, \quad \omega_1^3 = -\frac{2}{R}\omega^1, \quad \omega_2^3 = -\frac{2}{R}\omega^2,$$
$$\omega_1^2 = \omega_3^4 = 0,$$

and then $H = -\frac{2}{R}e_3 - \frac{2}{R}e_4$.

To the curvature lines on the torus correspond on the middle surface the curvature lines (circle and straight lines). We have then for the normals to the torus a congruence of Ribaucour and also a Ribaucour correspondence on the sheets of the associated sphere congruence which are the torus and a sphere of radius r with center at the origin. We can verify also that $K_N = 0$.

4. Marginally trapped submanifolds of codimension 2 in M^{n+2}

With the same method we can study marginally trapped submanifolds of codimension 2 in M^{n+2} as it was done in [11].

In this case the focal curve K is an algebraic curve of degree n whose equation is with the same notation

$$\begin{vmatrix} 1-xh_{11}^{n+1}+yh_{11}^{n+2} & -xh_{12}^{n+1}+yh_{12}^{n+2} & \dots & -xh_{1n}^{n+1}+yh_{1n}^{n+2} \\ -xh_{12}^{n+1}+yh_{12}^{n+2} & 1-xh_{22}^{n+1}h_{22}^{n+2} & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ -xh_{1n}^{n+1}+yh_{1n}^{n+2} & \dots & \dots & 1-xh_{nn}^{n+1}+yh_{nn}^{n+2} \end{vmatrix} = 0.$$

We obtain for the families D_+ and D_- of isotropic lines in the normal plane with n focus lying on K and $n-1$ points on each isotropic line describing codimension 2 marginally trapped submanifolds.

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