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BIMINIMAL GENERAL HELIX IN THE HEISENBERG GROUP $Heis^3$

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Abstract. In this paper, we study biminimal curves and characterize non-geodesic biminimal general helix in the Heisenberg group $Heis^3$. We show that non-geodesic biminimal general helix are biharmonic curves. Morover, we obtain the position vectors of biminimal general helix in the Heisenberg group $Heis^3$.

1. Introduction

Let $f:(M,g)\to(N,h)$ be a smooth function between two Riemannian manifolds. Then f is said to be *harmonic* over compact domain $\Omega\subset M$ if it is a critical point of the energy

$$E(f)=\int_\Omega h(df,df)\,dv_g,$$

where dv_g is the volume form of M. From the first variation formula it follows that is harmonic if and only if its tension field $\tau(f)=\operatorname{trace}_g\nabla df$ vanishes.

The *bienergy* $E_2(f)$ of f over compact domain $\Omega\subset M$ is defined by

$$E_2(f)=\int_\Omega h(\tau(f),\tau(f))\,dv_g. \tag{1.1}$$

Using the first variational formula one sees that f is a biharmonic map if and only if its *bitension field* vanishes identically, i.e.,

$$\widetilde{\tau}(f):=-\triangle^f(\tau(f))-\operatorname{trace}_g R^N(df,\tau(f))df=0, \tag{1.2}$$

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where

$$\triangle^f = -\mathrm{trace}_g(\nabla^f)^2 = -\mathrm{trace}_g\left(\nabla^f\nabla^f - \nabla^f_{\nabla^M}\right) \tag{1.3}$$

is the Laplacian on sections of the pull-back bundle $f^{-1}(TN)$ and R^N is the curvature operator of (N,h) defined by

$$R(X,Y)Z = -[\nabla_X,\nabla_Y]Z + \nabla_{[X,Y]}Z.$$

An isometric immersion $f:(M,g)\to(N,h)$ is called a $\lambda-$biminimal immersion if it is a critical point of the functional:

$$E_{2,\lambda}(f) = E_2(f) + \lambda E(f) \quad , \lambda\in\mathbb{R}$$

with respect to all normal variations.

The Euler-Lagrange equation for $\lambda-$biminimal immersions is

$$\widetilde{\tau}(f)^{\perp} = \lambda\tau(f). \tag{1.4}$$

In particular, f is called a biminimal immersion if it is a critical point of the bienergy functional E_2 with respect to all normal variation with compact support. Here, a normal variation means a variation $\{f_t\}$ through $f=f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is normal to M.

The Euler-Lagrange equation of this variational problem is $\widetilde{\tau}(f)^{\perp}=0$. Here $\widetilde{\tau}(f)^{\perp}$ is the normal component of $\widetilde{\tau}(f)$.

In this paper, we study biminimal curves and we characterize non geodesic biminimal general helix in Heisenberg group $Heis^3$. Morover, we obtain the position vectors of a biminimal general helix in the Heisenberg group $Heis^3$.

2. Left invariant metric in the Heisenberg group $Heis^3$

Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z) = (\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}). \tag{2.1}$$

$Heis^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g given by

$$g = dx^2+dy^2+(dz+\frac{y}{2}dx-\frac{x}{2}dy)^2. \tag{2.2}$$

The Lie algebra of $Heis^3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \tag{2.3}$$

for which we have the Lie products

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0\ ,$$

with

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above is given by:*

$$\nabla = \frac{1}{2}\begin{pmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & e_1 \\ -e_2 & e_1 & 0 \end{pmatrix}, \tag{2.4}$$

where the (i,j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k = 1, 2, 3\} = \{e_1, e_2, e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

the Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z, W)\,,$$

where X, Y, Z, W are smooth vector fields on $Heis^3$.

The components $\{R_{ijkl}\}$ of R relative to $\{e_1, e_2, e_3\}$ are defined by

$$g\left(R(e_i, e_j)e_k, e_l\right) = R_{ijkl}.$$

The non vanishing components of the above tensor fields are

$$R_{121} = -\tfrac{3}{4}e_2, \quad R_{131} = \tfrac{1}{4}e_3, \quad R_{122} = \tfrac{3}{4}e_1,$$

$$R_{232} = \tfrac{1}{4}e_3, \quad R_{133} = -\tfrac{1}{4}e_1, \quad R_{233} = -\tfrac{1}{4}e_2,$$

and

$$R_{1212} = -\tfrac{3}{4}, \quad R_{1313} = R_{2323} = \tfrac{1}{4}. \tag{2.5}$$

3. Biminimal curves in the Heisenberg group $Heis^3$

Let $\gamma : I \to Heis^3$ be a differentiable curve parametrized by arc length and let $\{T, N, B\}$ be the orthonormal frame field tangent to $Heis^3$ along γ and defined as follows: by T we denote the unit vector field γ' tangent to γ, by N the unit vector field in the direction of $\nabla_T T$ normal to γ, and we choose B so that $\{T, N, B\}$ is a positive oriented orthonormal basis. Then we have the following Frenet equations

$$\begin{aligned} \nabla_T T &= kN, \\ \nabla_T N &= -kT - \tau B, \\ \nabla_T B &= \tau N, \end{aligned} \tag{3.1}$$

where $k = |\tau(\gamma)| = |\nabla_T T|$ is the curvature of γ and τ its torsion. Expand T, N, B as

$$\begin{aligned} T =& T_1 e_1 + T_2 e_2 + T_3 e_3, \\ N =& N_1 e_1 + N_2 e_2 + N_3 e_3, \\ B =& B_1 e_1 + B_2 e_2 + B_3 e_3, \end{aligned}$$

with respect to the basis $\{e_1, e_2, e_3\}$.

We can write the tension field of γ as

$$\tau(\gamma) = \nabla_T T,$$

and the bitension field of γ as

$$\widetilde{\tau}(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T. \tag{3.2}$$

Theorem 3.1. *Let $\gamma : I \to Heis^3$ be a differentiable curve parametrized by arc length. Then is a non-geodesic biminimal curve if and only if*

$$k'' - k^3 - k\tau^2 = k(\frac{1}{4} - B_3^2), \tag{3.3}$$

$$2\tau k' + k\tau' = kN_3 B_3. \tag{3.4}$$

Proof. Using (3.1) and (3.2), we obtain

$$\begin{aligned} \widetilde{\tau}(\gamma) &= \nabla_T^3 T - kR(T, N)T \\ &= (-3kk')T + (k'' - k^3 - k\tau^2)N + (2\tau k' + k\tau')B - kR(T, N)T. \end{aligned}$$

From the vanishing of the normal components

$$\begin{aligned} k'' - k^3 - k\tau^2 - kR(T, N, T, N) = 0, \\ 2\tau k' + k\tau' - kR(T, N, T, B) = 0. \end{aligned} \tag{3.5}$$

On the other hand, using (2.5), we have

$$\begin{aligned} R(T,N,T,N) = &\sum_{i,j,l,p=1}^{3} T_l N_p T_i N_j R_{lpij} \\ =&T_1N_2T_1N_2R_{1212} + T_1N_2T_2N_1R_{1221} \\ +&T_2N_1T_2N_1R_{2121} + T_2N_1T_1N_2R_{2112} \\ +&T_1N_3T_1N_3R_{1313} + T_1N_3T_3N_1R_{1331} \\ +&T_3N_1T_3N_1R_{3131} + T_3N_1T_1N_3R_{3113} \\ +&T_2N_3T_2N_3R_{2323} + T_2N_3T_3N_2R_{2332} \\ +&T_3N_2T_3N_2R_{3232} + T_3N_2T_2N_3R_{3223} \\ -&\frac{3}{4}T_1^2N_2^2 + \frac{3}{4}T_1N_2T_2N_1 - \frac{3}{4}T_2^2N_1^2 + \frac{3}{4}T_2N_1T_1N_2 \\ +&\frac{1}{4}T_1^2N_3^2 - \frac{1}{4}T_1N_3T_3N_1 + \frac{1}{4}T_3^2N_1^2 - \frac{1}{4}T_3N_1T_1N_3 \\ +&\frac{1}{4}T_2^2N_3^2 - \frac{1}{4}T_2N_3T_3N_2 + \frac{1}{4}T_3^2N_2^2 - \frac{1}{4}T_3N_2T_2N_3 \\ =&-\frac{3}{4}(T_1N_2 - T_2N_1)^2 + \frac{1}{4}(T_1N_3 - T_3N_1)^2 \\ +&\frac{1}{4}(T_2N_3 - T_3N_2)^2. \end{aligned}$$

From $B_1^2 + B_2^2 + B_3^2 = 1$ and the fact that $B = T \times N$, we obtain

$$R(T,N,T,N) = \frac{1}{4} - B_3^2.$$

Similarly, we have

$$R(T,N,T,B) = N_3B_3.$$

These, together with (3.5), complete the proof of the theorem. □

The equations (3.3) and (3.4) can be deduced from ([1], (3.5)). Note that in [4], biminimal Legendre curves in $Heis^3$ are studied.

4. Biminimal general helix in the Heisenberg group $Heis^3$

Definition 4.1. (*see, for example,* [8]) Let $\gamma : I \to Heis^3$ be a curve of Heisenberg group $Heis^3$ and $\{T, N, B\}$ be a Frenet frame on $Heis^3$ along γ. If k and τ are positive constant along γ, then γ is called a helix with respect to Frenet frame.

Definition 4.2. (*see, for example,* [8]) Let $\gamma : I \to Heis^3$ be a curve of Heisenberg group $Heis^3$ and $\{T, N, B\}$ be a Frenet frame on $Heis^3$ along γ. A curve γ such that

$$\frac{k}{\tau} = \text{constant}$$

is called a general helix with respect to Frenet frame.

Theorem 4.1. *Let $\gamma : I \to Heis^3$ be a non-geodesic biminimal general helix parametrized by arc length if $N_3 B_3 = constant$, then γ is a helix.*

Proof. We can use (3.1) to compute the covariant derivatives of the vector fields T, N, B as:

$$\begin{aligned}
\nabla_T T =& (T_1' + T_2 T_3)e_1 + (T_2' + 2T_1 T_3)e_2 + T_3' e_3,\\
\nabla_T N =& (N_1' + \frac{1}{2}(T_2 N_3 + T_3 N_2))e_1 + (N_2' - \frac{1}{2}(T_1 N_3 + T_3 N_1))e_2\\
&+ (N_3' + \frac{1}{2}(T_1 N_2 - T_2 N_1))e_3,\\
\nabla_T B =& (B_1' + \frac{1}{2}(T_2 B_3 + T_3 B_2))e_1 + (B_2' - \frac{1}{2}(T_1 B_3 + T_3 B_1))e_2\\
&+ (N_3' + \frac{1}{2}(T_1 N_2 - T_2 N_1))e_3.
\end{aligned} \tag{4.1}$$

It follows that the first components of these vectors are given by

$$\begin{aligned}
< \nabla_T T, e_3 >&= T_3',\\
< \nabla_T N, e_3 >&= N_3' + \frac{1}{2}(T_1 N_2 - T_2 N_1),\\
< \nabla_T B, e_3 >&= B_3' + \frac{1}{2}(T_1 B_2 - T_2 B_1).
\end{aligned} \tag{4.2}$$

On the other hand, using Frenet formulas (3.1), we have,

$$\begin{aligned}
< \nabla_T T, e_3 >&= kN_3,\\
< \nabla_T N, e_3 >&= -kT_3 - \tau B_3,\\
< \nabla_T B, e_3 >&= \tau N_3.
\end{aligned} \tag{4.3}$$

These, together with (4.2) and (4.3), give

$$\begin{aligned}
T_3' =& kN_3,\\
N_3' + \frac{1}{2}B_3 =& - kT_3 - \tau B_3,\\
B_3' + \frac{1}{2}N_3 =& \tau N_3.
\end{aligned} \tag{4.4}$$

From (4.4), we have

$$B_3' = (\tau - \frac{1}{2})N_3. \tag{4.5}$$

Suppose that γ is a be a non-geodesic biminimal general helix with respect to the Frenet frame $\{T, N, B\}$. Then,

$$\frac{k}{\tau} = c. \tag{4.6}$$

Using (4.6), we have

$$k'\tau = \tau' k. \tag{4.7}$$

We substitute (4.7) in (3.4), we obtain

$$k' = \frac{1}{3}N_3B_3, \quad \tau' = \frac{c}{3}N_3B_3. \tag{4.8}$$

From N_3B_3 =constant it follows that

$$k'' = 0. \tag{4.9}$$

We substitute (4.9) in (3.3), we obtain

$$k^2 + \tau^2 = B_3^2 - \frac{1}{4}. \tag{4.10}$$

Next we replace $\tau = k/c$ in (4.10)

$$k^2 = \frac{1}{1 + \frac{1}{c^2}}(B_3^2 - \frac{1}{4}). \tag{4.11}$$

If (4.11) derived and taking into account (4.5) and (4.8), becomes

$$k = \frac{3(\tau - \frac{1}{2})}{1 + \frac{1}{c^2}}. \tag{4.12}$$

Substituting (4.6) in (4.12), we have

$$k = \text{constant}. \tag{4.13}$$

From (4.6), we obtain

$$\tau = \text{constant},$$

which implies γ circular helix. □

Theorem 4.2. *Let $\gamma : I \to Heis^3$ be a non-geodesic biminimal general helix parametrized by arc length and $N_3 B_3 = constant$. Then the position vector of the curve $\gamma(s)$*

$$\begin{aligned}\gamma(s) &= \left(s + C_1 k \sin\left(s\sqrt{B_3^2 - \tfrac{1}{4}}\right) - C_2 k \cos\left(s\sqrt{B_3^2 - \tfrac{1}{4}}\right) - \tfrac{ks}{B_3^2 - \frac{1}{4}}\right) T(s) \\ &+ \left(c_1 \cos\left(s\sqrt{B_3^2 - \tfrac{1}{4}}\right) + c_2 \sin\left(s\sqrt{B_3^2 - \tfrac{1}{4}}\right) - \tfrac{k}{B_3^2 - \frac{1}{4}}\right) N(s) \\ &+ \left(C_1 \tau \sin\left(s\sqrt{B_3^2 - \tfrac{1}{4}}\right) - C_2 \tau \cos\left(s\sqrt{B_3^2 - \tfrac{1}{4}}\right) - \tfrac{k\tau s}{B_3^2 - \frac{1}{4}}\right) B(s),\end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$, $C_1 = \frac{c_1}{\sqrt{B_3^2 - \frac{1}{4}}}$ and $C_2 = \frac{c_2}{\sqrt{B_3^2 - \frac{1}{4}}}$.

Proof. If $\gamma(s)$ is a non-geodesic biharmonic curve. Then we can write its position vector as follows:

$$\gamma(s) = \xi(s) T(s) + \eta(s) N(s) + \rho(s) B(s) \tag{4.14}$$

for some differentiable functions ξ, η and ρ of $s \in I \subset \mathbb{R}$. These functions are called component functions (or simply components) of the position vector.

Differentiating (4.14) with respect to s and by using the corresponding Frenet equation (3.1), we find

$$\begin{aligned}\xi'(s) - \eta(s) k &= 1, \\ \eta'(s) + \xi(s) k + \rho(s) \tau &= 0, \\ \rho'(s) - \eta(s) \tau &= 0.\end{aligned} \tag{4.15}$$

From (4.15) we get the following differential equation:

$$\eta''(s) + (k^2 + \tau^2)\eta(s) + k = 0. \tag{4.16}$$

By using (3.3) we find

$$\eta''(s) + (\frac{1}{4} - B_3^2)\eta(s) + k = 0. \tag{4.17}$$

The solution of (4.17) is

$$\eta(s) = c_1 \cos\left(s\sqrt{B_3^2 - \frac{1}{4}}\right) + c_2 \sin\left(s\sqrt{B_3^2 - \frac{1}{4}}\right) - \frac{k}{B_3^2 - \frac{1}{4}}, \tag{4.18}$$

where $c_1, c_2 \in \mathbb{R}$.

From $\xi'(s) = \eta(s)k + 1$ and using (4.18) we find the solution of this equation as follows:

$$\begin{aligned}\xi(s) =& s + \frac{c_1 k}{\sqrt{B_3^2 - \frac{1}{4}}} \sin\left(s\sqrt{B_3^2 - \frac{1}{4}}\right) \\ & - \frac{c_2 k}{\sqrt{B_3^2 - \frac{1}{4}}} \cos\left(s\sqrt{B_3^2 - \frac{1}{4}}\right) - \frac{ks}{B_3^2 - \frac{1}{4}}.\end{aligned} \tag{4.19}$$

By using (4.18) we find the solution of $\rho'(s) = \eta(s)\tau$ as follows:

$$\begin{aligned}\rho(s) =& \frac{c_1 \tau}{\sqrt{B_3^2 - \frac{1}{4}}} \sin\left(s\sqrt{B_3^2 - \frac{1}{4}}\right) \\ & - \frac{c_2 \tau}{\sqrt{B_3^2 - \frac{1}{4}}} \cos\left(s\sqrt{B_3^2 - \frac{1}{4}}\right) - \frac{k\tau s}{B_3^2 - \frac{1}{4}}.\end{aligned} \tag{4.20}$$

Substituting (4.18), (4.19) and (4.20) in (4.14) complete the proof of the theorem. □

Corollary 4.1. *If γ is non-geodesic biminimal general helix and $N_3 B_3 =$ constant, γ is a biharmonic curve.*

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