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DESCRIPTIONS OF ZERO SETS AND PARAMETRIC REPRESENTATIONS OF CERTAIN ANALYTIC AREA NEVANLINNA TYPE CLASSES IN THE UNIT DISK

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Abstract. A complete zero set description of a scale of analytic area Nevanlinna type spaces in the unit disk and parametric representations of these spaces are established. These generalize some well-known, classical results.

1. Definitions, Preliminaries and Problem Statement

Assuming that $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk of the finite complex plane $\mathbb{C}$, T is the boundary of $\mathbb{D}$ and $H(\mathbb{D})$ is the space of all functions holomorphic in $\mathbb{D}$, introduce the classes of functions:

$$\widetilde{N}^{\infty}_{\alpha} = \Big\{ f \in H(\mathbb{D}) : T(\tau, f) \leq C_f(1-\tau)^{-\alpha}, \ \ 0 \leq \tau \leq 1 \Big\}, \quad \alpha \geq 0,$$

where $T(\tau, f) = (1/2\pi) \int_T \log^+ |f(r\xi)| d\xi$ is Nevanlinna's characteristic (see eg. [10]). It is obvious that if $\alpha = 0$, then $\widetilde{N}^{\infty}_0 = N$, where N is Nevanlinna's class. The following statement holds by Nevanlinna's classical result on the parametric representation of N (see eg. [11]). *The analytic subset of N coincides with the set of functions representable in the form*

$$f(z) = C_\lambda z^\lambda B(z, \{z_k\}) \exp\left\{ \int_{-\pi}^{\pi} \frac{d\mu(\theta)}{1 - ze^{-i\theta}} \right\}, \quad z \in \mathbb{D},$$

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where C_λ is any complex number, λ is any nonnegative integer, $B(z,\{z_k\})$ is the classical Blaschke product with zeros $\{z_k\}_k \subset \mathbb{D}$ enumerated according their multiplicities and satisfying the condition $\sum_k(1-|z_k|) < +\infty$ and $\mu(\theta)$ is any function of bounded variation in $[-\pi,\pi]$.

In [3], the following proposition is established (see also [2]) for the sequences $\{z_k\}_{k=1}^\infty \subset \mathbb{D}$ satisfying the greater density condition

$$\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} < +\infty, \quad t > -1. \tag{1.1}$$

Proposition A. *If* (1.1) *is true for some $t > -1$, then the infinite product*

$$\Pi_t(z,\{z_k\}) = \prod_{k=1}^{+\infty}\left(1-\frac{z}{z_k}\right)\exp\left\{-\frac{t}{\pi}\int_{\mathbb{D}}\frac{(1-|\xi|^2)^t\ln\left|1-\frac{\xi}{z_k}\right|}{(1-\overline{\xi}z)^{t+2}}dm_2(\xi)\right\}, \tag{1.2}$$

where $m_2(\xi)$ is Lebesgue's area measure, converges absolutely and uniformly inside $\mathbb{D}$, where it presents an analytic function with zeros $\{z_k\}_{k=1}^\infty$.

We shall be based also on some other known statements which we give below. The next lemma can be found in [2].

Lemma A. *If* (1.1) *is true for some $t > -1$, then the following estimate holds for Djrbashian's product:*

$$\ln^+\left|\Pi_t(z,z_k)\right| \le C_t\sum_{k=0}^{+\infty}\left(\frac{1-|z_k|^2}{|1-z\overline{z_k}|}\right)^{t+2},$$

where $C_t > 0$ is a constant depending solely on t.

To state a theorem recently established in [11], by $B_\gamma^{p,q}(\mathrm{T})$, ($0 < p < \infty$, $0 < q \le \infty$, $\gamma > 0$) we denote the standard Besov class on the unit circle $\mathrm{T} = \{z : |z| = 1\}$ (see, eg. [1, 2, 10, 11]).

We will need Besov spaces on the unit circle which we will define with the help of Besov spaces on the real line. The Besov space $B_s^{p,q}(\mathbb{R})$, $0 < p < \infty$, $0 < q \le \infty$ is a complete quasinormed space which is a Banach space when $1 \le p, q \le \infty$. Let $\triangle_h f(x) = f(x-h) - f(x)$ and the modulus of continuity is defined by

$$\omega_p^2(f,t) = \sup_{|h|\le t}\|\triangle_h^2 f\|_p.$$

Let $n = 0, 1, 2, \ldots, s = n + \alpha$ with $0 < \alpha \leq 1$, the Besov space $B_s^{p,q}(\mathbb{R})$ contains all functions f such that $f \in W_p^n(\mathbb{R})$, where $W_p^n(\mathbb{R})$, $0 < p < \infty$, $n \in \mathbb{N}$ is a classical Sobolev space and

$$\int_0^\infty \left| \frac{\omega_p^2(f^{(n)}, t)}{t^\alpha} \right|^q \frac{dt}{t} < \infty.$$

The Besov space $B_s^{p,q}(\mathbb{R})$, $1 \leq p, q \leq \infty$ is equipped with the norm

$$\|f\|_{W_p^n(\mathbb{R})} + \left(\int_0^\infty \left| \frac{\omega_p^2(f^{(n)}, t)}{t^\alpha} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

To define the Besov space on unit circle we have to use the previous definition of Besov space on the real line and the standard map $t \to e^{it}$, $t \in \mathbb{R}$.

Theorem A. *If $\alpha \geq 0$ and $\beta > \alpha - 1$, then the class $\widetilde{N}_\alpha^\infty$ coincides with the set of functions representable in the form*

$$f(z) = C_\lambda z^\lambda \Pi_\beta(z, z_k) \exp\left\{ \int_{-\pi}^{\pi} \frac{\psi(e^{i\theta}) d\theta}{(1 - e^{-i\theta} z)^{\beta+2}} \right\}, \quad z \in \mathbb{D},$$

where C_λ is a complex number, λ is a nonnegative integer, $\Pi_\beta(z, \{z_k\})$ is Djrbashian's product (1.2), *$\{z_k\}_{k=1}^\infty \subset \mathbb{D}$ is a sequence satisfying the condition*

$$n(\tau) = card\,\{z_k : |z_k| < \tau\} \leq \frac{C}{(1 - \tau)^{\alpha+1}}$$

and $\psi(e^{i\theta})$ is a real function of $B_{\beta-\alpha+1}^{1,\infty}$.

We also give the below theorem which is established in [10] and in a sense is similar to Theorem A.

Theorem B. *If $0 < p < \infty$, $\alpha > -1$ and $\beta > (\alpha + 1)/p$, then*

$$\int_0^1 (1 - \tau)^\alpha T^p(\tau, f) d\tau < +\infty$$

if and only if

$$f(z) = C_\lambda z^\lambda \Pi_\beta(z, z_k) \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\psi(e^{i\theta}) d\theta}{(1 - e^{-i\theta} z)^{\beta+1}} \right\}, \quad z \in \mathbb{D},$$

where C_λ is any complex number, $\lambda \geq 0$ is any integer, $\{z_k\} \in \mathbb{D}$ is a sequence such that

$$\int_0^1 (1 - \tau)^{\alpha+p} n(\tau)^p d\tau < \infty,$$

and $\psi \in B_s^{1,p}(\mathrm{T})$, where $s = \beta - (\alpha + 1)/p$.

One can see that Theorem A gives the parametric representations of the spaces $\widetilde{N}_\alpha^\infty$, while Theorem B gives the parametric representations of some other analytic area Nevanlinna type spaces in the unit disk. One of the goals of this paper are the parametric representations of the larger spaces

$$N_{\alpha,\beta}^p = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\int_{|z|\le R} \ln^+ |f(z)|(1-|z|)^\alpha dm_2(z) \right]^p (1-R)^\beta dR < +\infty \right\},$$

$$N_{\alpha,\beta_1}^\infty = \left\{ f \in H(\mathbb{D}) : \sup_{0\le R<1} \left[\int_{|z|\le R} \ln^+ |f(z)|(1-|z|)^\alpha dm_2(z) \right] (1-R)^{\beta_1} < +\infty \right\},$$

where it is assumed that $\beta_1 \ge 0$, $\alpha > -1$, $\beta > -1$ and $0 < p < \infty$. Note that various properties of $N_{\alpha,0}^\infty$ are studied in [2]. In particular, the works [2, 10, 11] give complete description of zero sets and parametric representations of $N_{\alpha,0}^\infty$. Thus, it is natural to consider the problem *on extension of these important results to all* $N_{\alpha,\beta_1}^\infty$ *classes.*

The zero set description problem can be stated in the following simple form: *assuming that X is a fixed subspace of $H(\mathbb{D})$, precisely find a class Y of sequences such that the zero set of any function $f \in X$ is of Y and for any $\{z_k\}_{k=1}^\infty \in Y$ there is a function $f \in X$ satisfying $f(z_k) = 0$ $(k = 1, \dots, n)$* (see [2, 7]). Note that for many classical analytic classes, such as the space A_α^p, this problem is still open (see [2]). On the other hand, the complete descriptions of the zero sets of $N_{\alpha,0}^\infty$ and $\widetilde{N}_\alpha^\infty$ are known (see [2, 9, 10]). One of the intentions of this paper is to solve this problem for some new Nevanlinna type analytic classes in the unit disk and to establish the parametric representations of these classes, where the found description is used. We mention that several new results of this type for some classical Nevanlinna-Djrbashian analytic classes in the unit disk are presented in [2, 10, 11]. So, it is natural to consider the problem for $N_{\alpha,\beta}^p$ and $N_{\alpha,\beta_1}^\infty$.

Everywhere below, by $n(t) = n_f(t)$ we denote the quantity of zeros of an analytic function f in the disk $|z| \le t < 1$ and by $\mathcal{Z}(X)$ the zero set of an analytic class X, $X \subset H(D)$. Besides, we assume that

$$(NA)_{p,\gamma,v} = \left\{ f \in H(\mathbb{D}) : \int_0^1 \left[\sup_{0<\tau<R} T(f,\tau)(1-\tau)^\gamma \right]^p (1-R)^v dR < +\infty \right\},$$

where $\gamma \ge 0$, $v > -1$ and $0 < p < \infty$, and

$$N_{\alpha,\beta}^{\infty,p} = \left\{ f \in H(\mathbb{D}) : \sup_{0\le R<1} \int_0^R \left[\int_{\mathrm{T}} \ln^+ |f(|z|\xi)| d\xi \right]^p (1-|z|)^\alpha d|z| (1-R)^\beta < +\infty \right\},$$

where $0 < p < \infty$, $\alpha > -1$ and $\beta \geq 0$. Note that the zero sets of $N^{\infty;p}_{\alpha,\beta}$ are described in [10] for $\beta = 0$.

It is not difficult to verify that all the above mentioned analytic classes are topological vector spaces with complete invariant metrics. We note that the mentioned problems of zero set description and parametric representation have various applications and are important in function theory (see, eg. [4, 5, 6]). Solution of many problems, for instance on existence of radial limits, is based on descriptions of zero sets and parametric representations. Zero set descriptions and parametric representations are used also in spectral theory of linear operators (see, eg. [8]). All results of this paper are given without proofs in our recent note [12]. The authors intend to publish separate papers with various applications of these results to the description problem of closed ideals in area Nevanlinna spaces over the unit disk and to some other problems in these spaces (see, eg. [13]).

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

2. Theorems on Zero Sets of $N^p_{\alpha,\beta}$, $(NA)_{p,\gamma,v}$ and $N^{\infty;p}_{\alpha,\beta}$ Classes

This section gives the descriptions of the zero sets of the defined above area Nevanlinna type classes.

Theorem 2.1. *For any numbers $0 < p < \infty$, $\alpha > -1$ and $\beta > -1$, the following conditions are equivalent.*

$$\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(2p+1)} 2^{k\alpha p} 2^{k\beta}} < +\infty, \tag{2.1}$$

and

$$\{z_k\} \in \mathcal{Z}\left(N^p_{\alpha,\beta}\right),$$

where $n_k = n(1 - 2^{-k})$ and $n(\tau) = card\,\{z_k : |z_k| < \tau\}$. If (2.1) is true, then

$$\Pi_t(z, z_k) \in N^p_{\alpha,\beta} \quad for \quad t > \max\left[(\alpha + \beta/p) + \max\{1, 1/p\}, (\alpha + 1)\right].$$

Theorem 2.2. *For any numbers $0 < p < \infty$, $v \geq 0$ and $\gamma \geq 0$, the following conditions are equivalent*

$$\{z_k\} \in \mathcal{Z}\Big((NA)_{p,\gamma,v}\Big),$$

$$\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k[(p\gamma+1)+v+1]}}<+\infty, \tag{2.2}$$

where $n_k = n(1-2^{-k})$. *If* (2.2) *is true, then*

$$\Pi_t(z,z_k)\in(NA)_{p,\gamma,v}\ \ \text{for}\ \ p\le 1,\quad t>\frac{v+1}{p}+\gamma-1\ \ \text{and for}\ \ p>1,\ t>\frac{v}{p}+\gamma.$$

Theorem 2.3. *For any numbers* $0<p<\infty$, $\alpha\ge 0$ *and* $\beta>0$, *the following conditions are equivalent*

$$\{z_k\}_{k=1}^{\infty}\in \mathcal{Z}\left(N^{\infty,p}_{\alpha,\beta}\right);$$

$$n(\tau)\le C(1-\tau)^{-(\alpha+\beta+p+1)/p},\quad \tau\in(0,1). \tag{2.3}$$

If (2.3) *is true, then*

$$\Pi_t(z,z_k)\in N^{\infty,p}_{\alpha,\beta}\quad \text{for any}\quad t>\frac{\alpha+\beta+1}{p}-1.$$

Theorem 2.1 – 2.3 immediately give as corollaries the following parametric representations of mentioned above area Nevanlinna type classes.

Theorem 2.4. *If* $0<p<\infty$, $\alpha>-1$ *and* $\beta>-1$, *then the class* $N^p_{\alpha,\beta}$ *coincides with the set of functions representable for* $z\in\mathbb{D}$ *as*

$$f(z)=C_\lambda z^\lambda\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left\{\frac{t+1}{\pi}\int_0^1\!\!\int_{-\pi}^{\pi}\frac{(1-\rho^2)\ln\left|1-\frac{\rho e^{i\varphi}}{z_k}\right|}{(1-\rho e^{-i\varphi}z)^{t+2}}\rho d\rho d\varphi\right\}\exp\{h(z)\},$$

where $t>\max\left\{\left(\alpha+\beta/p\right)+\max\{1,1/p\},\,(\alpha+1)\right\}$, C_λ *is a complex number,* $\lambda\ge 0$,

$$\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k(\beta+2p+1+\alpha p)}}<+\infty,$$

and $h\in H(\mathbb{D})$ *is a function satisfying the condition*

$$\int_0^1\left[\int_0^R\left(\int_{-\pi}^{\pi}|h(\tau e^{i\varphi})|d\varphi\right)(1-\tau)^\alpha d\tau\right]^p(1-R)^\beta dR<+\infty.$$

Theorem 2.5. *If* $0<p<\infty$, $v\ge 0$ *and* $\gamma\ge 0$, *then the class* $(NA)_{p,\gamma,v}$ *coincides with the set of functions representable for* $z\in\mathbb{D}$ *as*

$$f(z)=C_\lambda z^\lambda\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)\exp\left\{\frac{t+1}{\pi}\int_0^1\!\!\int_{-\pi}^{\pi}\frac{(1-\rho^2)\ln\left|1-\frac{\rho e^{i\varphi}}{z_k}\right|}{(1-\rho e^{-i\varphi}z)^{t+2}}\rho d\rho d\varphi\right\}\exp\{h(z)\},$$

where for $p \leq 1$, $t > \frac{v+1}{p} + \gamma - 1$ *and for* $p > 1$, $t > \frac{v}{p} + \gamma$, C_λ *is a complex number,* $\lambda \geq 0$,

$$\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k[(p\gamma+1)+v+1]}} < +\infty,$$

and $h \in H(\mathbb{D})$ *is a function satisfying the condition*

$$\int_0^1 \left[\sup_{0<\tau<R} \int_{\mathrm{T}} |h(\tau\xi)| d\xi (1-\tau)^\gamma \right]^p (1-R)^v dR < +\infty.$$

Theorem 2.6. *If* $0 < p < \infty$, $\alpha \geq 0$ *and* $\beta > 0$, *then the class* $N_{\alpha,\beta}^{\infty,p}$ *coincides with the set of functions representable for* $z \in \mathbb{D}$ *as*

$$f(z) = C_\lambda z^\lambda \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\left\{ \frac{t+1}{\pi} \int_0^1 \int_{-\pi}^{\pi} \frac{(1-\rho^2) \ln\left|1 - \frac{\rho e^{i\varphi}}{z_k}\right|}{(1-\rho e^{-i\varphi} z)^{t+2}} \rho d\rho d\varphi \right\} \exp\{h(z)\},$$

where $n(\tau) \leq C(1-\tau)^{-(\alpha+\beta+p+1)/p}$, $\quad \tau \in (0,1)$, C_λ *is a complex number,* $\lambda \geq 0$, *and* $h \in H(\mathbb{D})$ *is a function satisfying the condition*

$$\sup_{0 \leq R < 1} \int_0^R \left[\int_{\mathrm{T}} |h(\tau\xi)| d\xi \right]^p (1-\tau)^\alpha d\tau (1-R)^\beta < +\infty.$$

We omit the proof of Theorem 2.4 – 2.6, since it immediately follows by the description of zero sets of the corresponding classes and some standard argument applied in [10] to some other analytic area Nevanlinna classes.

The given below proofs of Theorems 2.1, 2.2, 2.3 follow mainly by some standard arguments (see, eg. [2, 10, 11] and their references) based on Lemma A and the classical Jensen formula, but with more careful examination of estimates.

Proofs follow by the same scheme. First we use the classical Jensen inequality to show that the conditions (2.1), (2.2) and (2.3) are necessary. Then we prove the converse statements by application of Lemma A and Proposition A for great enough numbers t depending on α, β or v and γ.

Proof of Theorem 2.1. Let $f \in N_{\alpha,\beta}^p$. Then, without loss of generality it can be assumed that $f(0) = 1$, $f(z_k) = 0$ $(k = 1, 2, \ldots)$. Hence, by Jensen's inequality

$$I = \int_0^1 \left[\int_0^R (1-\tau)^\alpha d\tau \int_0^\tau \frac{n(u)}{u} du \right]^p (1-R)^\beta dR$$
$$\leq C_2 \int_0^1 \left[\int_{|z|<R} \log^+ |f(z)| (1-|z|)^\alpha dm_2(z) \right]^p (1-R)^\beta dR.$$

Further, it is obvious that

$$\int_0^{\tau} \frac{n(u)}{u} du \geq \int_{\tau-\frac{R-\tau}{2}}^{\tau} \frac{n(u)}{u} du \geq C_2\, n\left(\frac{3\tau - R}{2}\right) \frac{R-\tau}{2},$$

and

$$\|f\|^p_{N^p_{\alpha,\beta}} \geq C_2 \int_{C_1}^{1} \left[\int_C^R (R-\tau)^{\alpha+1} n\left(\frac{3\tau-R}{2}\right) d\tau\right]^p (1-R)^{\beta} dR$$

for any numbers $R < 3\tau$, $C_1 > C$, $C < R$, $C_1 < R < 1$, C, $C_1 > 0$, $\alpha > 0$. Besides, one can see that the following implications are true:

$$\frac{3\tau - R}{2} = \rho \;\Rightarrow\; \tau = \frac{2\rho + R}{3} \;\Rightarrow\; R - \tau = \frac{2(R-\rho)}{3}.$$

Hence,

$$\int_C^R (R-\tau)^{\alpha+1} n\left(\frac{3\tau - R}{2}\right) d\tau \geq C_2 \int_{(3C-R)/2}^{R} n(\rho)(R-\rho)^{\alpha+1} d\rho.$$

Suppose $C = (4R-1)/3$. Then $(3C-R)/2 = R - (1-R)/2$ and

$$\begin{aligned}
\|f\|^p_{N^p_{\alpha,\beta}} &\geq C_2 \int_{C_1}^{1} \left[\int_{R-\frac{1-R}{2}}^{R} n(\rho)(R-\rho)^{\alpha+1} d\rho\right]^p (1-R)^{\beta} dR \\
&\geq C_2 \int_{C_1}^{1} \left[n\left(\frac{3R-1}{2}\right)\right]^p (1-R)^{(\alpha+1)p+\beta+p} dR \\
&\geq C_2 \int_{C_1^*}^{1} [n(\rho)]^p (1-\rho)^{(\alpha+1)p+\beta+p} d\rho \asymp \sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(p+1)} 2^{k(\alpha+1)p+k\beta}},
\end{aligned}$$

since

$$\int_0^1 f(\rho) d\rho = \sum_{k=1}^{\infty} \int_{\tau_k}^{\tau_{k+1}} f(\tau) d\tau$$

for any $f \in L^1(0,1)$ and $\tau_k = 1 - \frac{1}{2^{k+1}}$ $(k = 0, 1, 2, \ldots)$ and

$$n(s_1) \leq n(s_2) \quad \text{when} \quad 0 \leq s_1 \leq s_2 < 1.$$

For $\alpha \in (-1, 0]$, a similar argument leads to the estimate

$$\begin{aligned}
\|f\|^p_{N^p_{\alpha,\beta}} &\geq C_2 \int_{C_1}^{1} \left[\int_{R-\frac{1-R}{2}}^{R} n(\rho)(R-\rho)\left(\frac{3-2\rho-R}{3}\right)^{\alpha} d\rho\right]^p (1-R)^{\beta} dR \\
&\geq C_2 \int_{C_1}^{1} \left[n\left(\frac{3R-1}{2}\right)\right]^p (1-R)^{(\alpha+1)p+\beta+p} dR.
\end{aligned}$$

Then, we continue as in the above case $\alpha > 0$ and come to the desired statement.

For proving the converse statement, fix a number t so that Lemma A and Proposition A are applicable. Further, observe that $|\log|f||$ and $\log^+|f|$ both belong to $N^p_{\alpha,\beta}$ if just one of them is of $N^p_{\alpha,\beta}(\mathbb{D})$. Hence, for $z=\tau e^{i\varphi}$, $\tau=t+2$ we get

$$\int_{-\pi}^{\pi}\Big|\log\left|\Pi_t\left(z,\{z_k\}\right)\right|\Big|d\varphi \le C\sum_{k=1}^{\infty}(1-|z_k|)^{t+2}\int_{-\pi}^{\pi}\frac{d\varphi}{|1-z_ke^{i\varphi}|^{\tau}}.$$

Hence, for great enough values of t

$$\begin{aligned}\int_0^R T(\Pi_t,\rho)(1-\rho)^{\alpha}d\rho &\le C\int_0^R\sum_{k=1}^{\infty}\frac{(1-|z_k|)^{t+2}}{(1-z_k\rho)^{t+1}}(1-\rho)^{\alpha}d\rho\\ &\le C\int_0^R(1-\rho)^{\alpha}\int_0^1\frac{(1-s)^{t+2}}{(1-s\rho)^{t+1}}dn(s)d\rho=J(R,f).\end{aligned}$$

Therefore,

$$\int_0^1\frac{(1-s)^{t+2}}{(1-s\tau)^{t+1}}dn(s)\le C\int_0^1\frac{(1-s)^{t+1}}{(1-s\tau)^{t+1}}n(s)ds,$$

and hence

$$\int_0^1\frac{(1-s)^{t+1}n(s)}{(1-s\rho)^{t+1}}ds\le C\sum_{k=1}^{\infty}\frac{n_k}{2^{k(t+2)}}\frac{1}{(1-\tau_k\rho)^{t+1}}.$$

Consequently, for $p\le 1$

$$J\le C\sum_{k=1}^{\infty}\int_0^R\frac{(1-\rho)^{\alpha}d\rho}{(1-\tau_k\rho)^{t+1}}\frac{n_k}{2^{k(t+2)}}\le C\sum_{k=1}^{\infty}\frac{n_k}{2^{k(t+2)}}\frac{1}{(1-\tau_kR)^{(t+1)-(\alpha+1)}},$$

and by the inequality $\left[\sum_{k=1}^{\infty}a_k\right]^p\le\sum_{k=1}^{\infty}a_k^p$ $(p\le 1)$ we get

$$\int_0^1 J^p(f,R)(1-R)^{\beta}dR\le C\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k(2p+1+\alpha p+\beta)}}.$$

If $p>1$, then the following estimates are true:

$$\begin{aligned}\int_0^1 J^p(f,R)(1-R)^{\beta}dR &\le C\int_0^1\left[\sum_{k=1}^{\infty}\int_0^R\frac{(1-\rho)^{\alpha}d\rho}{(1-\tau_k\rho)^{t+1}}\frac{n_k}{2^{k(t+2)}}\right]^p(1-R)^{\beta}dR\\ &\le C\int_0^1\left[\sum_{k=1}^{\infty}\frac{n_k}{2^{k(t+2)}}\frac{1}{(1-\tau_kR)^{(t+1-(\alpha+1))}}\right]^p(1-R)^{\beta}dR.\end{aligned}$$

Or, which is the same,

$$
\begin{aligned}
M &= \int_0^1 J^p(f,R)(1-R)^\beta dR \\
&\preceq \int_0^1 (1-R)^\beta \left[\int_0^R (1-\rho)^\alpha \int_0^1 \frac{(1-s)^{t+2}}{(1-s\rho)^{t+1}} dn(s) d\rho \right]^p dR \\
&\preceq \int_0^1 (1-R)^\beta \left[\int_0^R (1-\rho)^\alpha \int_0^1 \frac{(1-s)^{t+1}}{(1-s\rho)^{t+1}} n(s) ds d\rho \right]^p dR \\
&= \int_0^1 (1-R)^\beta \left[\int_0^R (1-\rho)^\alpha \sum_{k=1}^\infty \int_{1-2^{-k}}^{1-2^{-(k+1)}} \frac{(1-s)^{t+1}}{(1-s\rho)^{t+1}} n(s) ds d\rho \right]^p dR \\
&\preceq \int_0^1 (1-R)^\beta \left[\int_0^R \sum_{k=1}^\infty n_k \frac{2^{-k(t+2)}(1-\rho)^\alpha d\rho}{(1-\rho\tau_k)^{t+1}} \right]^p dR \\
&\preceq \int_0^1 (1-R)^\beta \left[\sum_{k=1}^\infty n_k 2^{-k(t+2)} \frac{1}{(1-R\tau_k)^{t-\alpha}} \right]^p dR, \quad t > \alpha.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
M &\preceq \int_0^1 (1-R)^\beta \left[\int_0^1 \frac{n(\rho)(1-\rho)^{t+1}}{(1-\rho R)^{t-\alpha}} d\rho \right]^p dR \\
&= \int_0^1 (1-R)^{\beta/p} \left(\int_0^R + \int_R^1 \right) \psi(R) dR = I_1 + I_2
\end{aligned}
$$

for any function $\psi \geq 0$ such that $\|\psi\|_{L^q} = 1$ $(1/p + 1/q = 1)$. Using the Hardy and Hölder inequalities, one can be convinced that

$$
\begin{aligned}
I_1 &= \int_0^1 n(\rho)(1-\rho)^{t+1} \int_0^\rho \frac{(1-R)^{\beta/p}\psi(R)}{(1-\rho R)^{t-\alpha}} dR d\rho \\
&\preceq \int_0^1 n(\rho)(1-\rho)^{t+1+\frac{\beta}{p}+\alpha-t+1} \int_0^\rho \frac{\psi(R)}{1-R} dR d\rho,
\end{aligned}
$$

$$
\begin{aligned}
I_1 &\leq \int_0^1 \frac{\psi(\tau)}{1-\tau} \int_\tau^1 n(\rho)(1-\rho)^{\frac{\beta}{p}+\alpha+2} d\rho d\tau \\
&\leq \left(\int_0^1 \psi^q(\tau) d\tau \right)^{\frac{1}{q}} \cdot \left(\int_0^1 \left(\frac{1}{1-\tau} \int_0^{1-\tau} n(1-t) t^{\frac{\beta}{p}+\alpha+2} dt \right)^p d\tau \right)^{\frac{1}{p}}
\end{aligned}
$$

and hence

$$
I_1 \preceq \int_0^1 n(\rho)^p (1-\rho)^{p+\beta+\alpha p+p} d\rho.
$$

For $\beta < 0$ above we used $(1-R)^{\frac{\beta}{p}} < (1-\rho)^{\frac{\beta}{p}}$, $R \leq \rho < 1$ for $\beta \geq 0$, $(1-R)^{\frac{\beta}{p}} < (1-\rho R)^{\frac{\beta}{p}}$, $\rho, R \in (0,1)$, for $t > \max\Big\{(\alpha + \beta/p) + \max\{1, 1/p\}, (\alpha+1)\Big\}$. Besides for

$\beta \geq 0$ again by Hölder and Hardy inequalities we will have

$$
\begin{aligned}
I_2 &= \int_0^1 n(\rho)(1-\rho)^{t+1} \int_\rho^1 \frac{(1-R)^{\beta/p}\psi(R)dR}{(1-\rho R)^{t-\alpha}} \\
&\preceq \int_0^1 n(\rho)(1-\rho)^{1+\frac{\beta}{p}+\alpha} \int_0^{1-\rho} \psi(1-u)du d\rho \\
&\preceq \left[\int_0^1 n(\rho)^p (1-\rho)^{p+\beta+\alpha p+p} d\rho\right]^{1/p} \left[\int_0^1 \left(\frac{1}{1-\rho}\int_0^{1-\rho} \psi(1-u)du\right)^q\right]^{1/q} \\
&\preceq B\cdot C\|\psi\|_{L^q}, \quad q>1.
\end{aligned}
$$

where

$$
B = \left[\int_0^1 n(\rho)^p (1-\rho)^{2p+\beta+\alpha p} d\rho\right]^{1/p} \asymp \left[\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(p+1)}2^{k(\alpha+1)p+k\beta}}\right]^{1/p}
$$

for $t > \max\Big\{(\alpha+\beta/p)+\max\{1,1/p\},\ (\alpha+1)\Big\}$.

The estimate of I_2 in case of $\beta<0$ needs small modification of mentioned arguments and we omit details.

Now we shall show that for great enough numbers t Lemma A and Proposition A are applicable. To this end, we prove that if $t > \max\Big\{(\alpha+\beta/p)+\max\{1,1/p\},\ (\alpha+1)\Big\}$, then $\sum_{k=1}^{\infty}(1-|z_k|)^{t+2}<\infty$. Hence, the condition

$$
\sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(\beta+\alpha p+2p+1)}} < \infty
$$

will imply the convergence of the product $\Pi_t(z,\{z_k\})$.

Indeed, the obvious inequality

$$
\int_0^1 n^p(\tau)(1-\tau)^{\beta+\alpha p+2p} d\tau < +\infty
$$

implies that

$$
\int_{\tau_1}^1 n^p(\tau)(1-\tau)^{\beta+\alpha p+2p} d\tau \to 0 \quad \text{as} \quad \tau_1 \to 1.
$$

Hence, for $\beta+\alpha p+2p>-1$

$$
n^p(\tau)(1-\tau)^{\beta+\alpha p+2p+1} \to 0 \quad \text{as} \quad \tau\to 1,
$$

and therefore $n(\tau) \leq C(1-\tau)^{-(\beta+\alpha p+2p+1)/p}$ $(0<\tau<1)$. Consequently,

$$\begin{aligned}\sum_{k=1}^{\infty}(1-|z_k|)^{t+2} &\leq C\sum_{k=1}^{\infty}\sum_{z_k\in B_k}(1-|z_k|)^{t+2}n_k\\ &\leq C\sum_{k=1}^{\infty}\sum_{z_k\in B_k}(1-|z_k|)^{t+2-(\beta+\alpha p+2p+1)/p}\\ &\leq C\sum_{k=1}^{\infty}\frac{1}{2^{k[t-(\beta+1)/p-\alpha]}}<+\infty\end{aligned}$$

where $B_k=\{z_k : |z_k|\in(\tau_k,\tau_{k+1})\}$ and $t>\max\Big\{(\alpha+\beta/p)+\max\{1,1/p\},\ (\alpha+1)\Big\}$. Thus, the latter requirement on t provides the convergence of the product $\Pi_t(z,\{z_k\})$, and the proof is complete. □

Proof of Theorem 2.2. We start as in Proof of Theorem 2.1. By the classical Jensen inequality (see, eg. [2])

$$\int_0^1\left[\sup_{0<\tau<R}\left(\int_0^{\tau}\frac{n(u)}{u}du\right)(1-\tau)^{\gamma}\right]^p(1-R)^v dR\leq C\|f\|^p_{(NA)_{p,\gamma,v}}.$$

Therefore, the following inequalities are true for any R, $\widetilde{R}\in(1/3,1)$ such that $\widetilde{R}=(3R-1)/2<R$:

$$\begin{aligned}\sup_{0<\tau<R}\int_0^{\tau}\frac{n(u)}{u}du(1-\tau)^{\gamma} &\geq C\sup_{1/3<\tau<R}\int_{\tau-\frac{1-\tau}{2}}^{\tau}\frac{n(u)}{u}du(1-\tau)^{\gamma}\\ &\geq C\sup_{1/3<\tau<R}n\left(\frac{3\tau-1}{2}\right)(1-\tau)(1-\tau)^{\gamma}\\ &\geq C\sup_{\rho\in(0,\widetilde{R})}n(\rho)(1-\rho)^{\gamma+1}\\ &\geq C\sup_{\rho\in(C,\widetilde{R})}n(\rho)(1-\rho)^{\gamma+1}\end{aligned}$$

Besides, one can see that for $\widetilde{\widetilde{R}}=\widetilde{R}-(1-\widetilde{R})/2$

$$\|f\|^p_{(NA)_{p,\gamma,v}}\geq C\int_{\tau_0}^1(1-R)^v\sup_{\rho\in(\widetilde{\widetilde{R}},\widetilde{R})}n(\rho)^p(1-\rho)^{(\gamma+1)p}dR$$

and

$$\begin{aligned}\|f\|^p_{(NA)_{p,\gamma,v}} &\geq C\int_{\tau_0}^1(1-\widetilde{R})^{p(\gamma+1)+v}n(\widetilde{R})^p dR\geq C\int_{\tau_0}^1 n(\widetilde{\widetilde{R}})^p(1-\widetilde{R})^{p(\gamma+1)+v}d\widetilde{R}\\ &\geq C\sum_{k=1}^{\infty}\frac{n_k^p}{2^{k[p(\gamma+1)+v+1]}},\quad n_k=n\left(1-2^{-k}\right),\quad k=0,1,2,3,\ldots.\end{aligned}$$

For proving the converse statement, we fix a number t such that Lemma A and Proposition A can be used. Then, we observe that for $p \le 1$

$$\|f\|^p_{(NA)_{p,\gamma,v}} = \int_0^1 \left[\sup_{0<\tau<R} T(\tau, f)(1-\tau)^\gamma \right]^p (1-R)^v dR$$

and

$$\int_{-\pi}^{\pi} \left| \log \left| \Pi_t(z, \{z_k\}) \right| \right| d\varphi \le C \sum_{k=1}^{\infty} (1-|z_k|)^{t+2} \int_{-\pi}^{\pi} \frac{d\varphi}{|1-\tau\tau_k e^{i\varphi}|^{t+2}},$$

where it is denoted $|z_k| = \tau_k$ and $z_k = \tau_k \xi_k$, $\tau_k = 1 - \frac{1}{2^{k+1}}$. Hence

$$\begin{aligned}
\|\Pi_t\|^p_{(NA)_{p,\gamma,v}} &\le C \int_0^1 \left[\sum_{k=1}^{\infty} \frac{(1-|z_k|)^{t+2}}{(1-R|z_k|)^{t+1-\gamma}} \right]^p (1-R)^v dR \\
&\le C \int_0^1 (1-R)^v \left[\int_0^1 \frac{(1-s)^{t+1} n(s) ds}{(1-Rs)^{t+1-\gamma}} \right]^p dR \\
&\le C \int_0^1 (1-R)^v \sum_{k=1}^{\infty} \frac{n_k^p 2^{-k[(t+1)p+p]}}{\left[1 - \left(1 - \frac{1}{2^{k+1}}\right) R\right]^{(t+1-\gamma)p}} dR \\
&\le C \sum_{k=1}^{\infty} n_k^p \frac{2^{-k(t+1)p} 2^{-kp}}{2^{-k[(t+1-\gamma)p-v-1]}} \le C \sum_{k=1}^{\infty} \frac{n_k^p}{2^{k[p(\gamma+1)+v+1]}}
\end{aligned}$$

for $t > (v+1)/p + \gamma - 1$, since one can easily verify that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(1-|z_k|)^{t+2}}{(1-R|z_k|)^{t+1-\gamma}} &= \int_0^1 \frac{(1-s)^{t+2} dn(s)}{(1-Rs)^{t+1-\gamma}} \\
&= \frac{(1-s)^{t+2} n(s)}{(1-Rs)^{t+1-\gamma}} \bigg|_0^1 - \int_0^1 n(s) \left(\frac{(1-s)^{t+2}}{(1-Rs)^{t+1-\gamma}} \right)'_s ds \\
&= -\int_0^1 n(s) \left[\frac{-(t+2)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}} + \frac{(1-s)^{t+2}(-(t+1-\gamma))}{(1-Rs)^{t+2-\gamma}} (-R) \right] ds \\
&= \int_0^1 \frac{n(s)(t+2)(1-s)^{t+1} ds}{(1-Rs)^{t+1-\gamma}} - \int_0^1 \frac{n(s)(1-s)^{t+2}}{(1-Rs)^{t+2-\gamma}} (t+1-\gamma) R ds \\
&\le C \int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}} (t+2) ds,
\end{aligned}$$

that

$$\begin{aligned}
\left[\int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}} ds \right]^p &\le C \left[\sum_{k=1}^{\infty} \frac{n\left(1-2^{-k-1}\right) 2^{-k(t+1)} 2^{-k}}{(1-\rho_k R)^{t+1-\gamma}} \right]^p \\
&\le C \sum_{k=1}^{\infty} \frac{n_k^p 2^{-k(t+1)p} 2^{-kp}}{(1-\rho_k R)^{(t+1-\gamma)p}}
\end{aligned}$$

and that for any $\tau = (t+1-\gamma)p-(v+1) > 0$ and $v > -1$

$$\int_0^1 \frac{(1-R)^v}{(1-\rho_k R)^{(t+1-\gamma)p}} dR \le C\left(\frac{1}{2^{-k}}\right)^\tau, \quad \rho_k = 1-\frac{1}{2^k} \quad (k=0,1,2,\dots).$$

Now, let $p > 1$. Then for the conjugate index $q > 1$ deduced by $1/p + 1/q = 1$ and any $t > \gamma - 1$

$$\begin{aligned}\left\|\Pi_t\right\|^p_{(NA)_{p,\gamma,v}} &\le C\int_0^1 \left[\int_0^1 \frac{n(s)(1-s)^{t+1}ds}{(1-Rs)^{t+1-\gamma}}\right]^p (1-R)^v dR\\ &= C\Big[I_1 + I_2\Big]\\ &= C\int_0^1 (1-R)^{\frac{v}{p}}\psi(R)\left(\int_0^1 \frac{n(s)(1-s)^{t+1}ds}{(1-Rs)^{t+1-\gamma}}\right)dR,\end{aligned}$$

where ψ is a nonnegative function such that $\|\psi\|_{L^q} = 1$,

$$\begin{aligned}I_1 &= \int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}}\left(\int_0^s \psi(R)(1-R)^{v/p}dR\right)ds\\ I_2 &= \int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-Rs)^{t+1-\gamma}}\left(\int_s^1 \psi(R)(1-R)^{v/p}dR\right)ds\end{aligned}$$

and

$$I_1 \le C\int_0^1 n(s)(1-s)^{t+1}\int_0^s \frac{\psi(R)(1-R)^{v/p}}{(1-R)^{t+1-\gamma}}dRds.$$

Further, by Hardy and Hölder inequalities

$$\begin{aligned}I_1 &\le C\int_0^1 \frac{n(s)(1-s)^{t+1}}{(1-s)^{t-\gamma-v/p}}\int_0^s \frac{\psi(R)}{1-R}dRds\\ &\le C\int_0^1 [n(s)]^p(1-s)^{\gamma p+p+v}ds\\ &\asymp \sum_{k=1}^{\infty} \frac{n_k^p}{2^{k(p(\gamma+1)+v+1)}}\end{aligned}$$

for $t > \gamma + v/p$. Besides, for $t > \gamma - 1$

$$\begin{aligned}
I_2 &= \int_0^1 n(s)(1-s)^{t+1} ds \int_s^1 \frac{(1-R)^{v/p}\psi(R)}{(1-sR)^{t+1-\gamma}} dR \\
&\le C \int_0^1 n(s) \frac{(1-s)^{t+1}}{(1-s)^{t+1}} \left(\int_s^1 \frac{(1-R)^{v/p}\psi(R)}{(1-sR)^{-\gamma}} dR \right) ds \\
&= \int_0^1 n(s) \left(\int_s^1 \frac{(1-R)^{v/p}\psi(R)}{(1-s)^{-\gamma}} dR \right) ds \\
&\le C \int_0^1 n(s)(1-s)^{\frac{v}{p}+\gamma} \int_0^{1-s} \psi(1-u) du ds \\
&\le C \left[\int_0^1 (n(s))^p (1-s)^{v+\gamma p+p} ds \right]^{1/p} \left[\int_0^1 \left(\frac{1}{1-s} \int_0^{1-s} \psi(1-u) du \right)^q ds \right]^{1/q} \\
&\le C\|\psi\|_{L^q} \left[\int_0^1 (n(s))^p (1-s)^{v+\gamma p+p} ds \right]^{1/p} \\
&\asymp C\|\psi\|_{L^q} \sum_{k=1}^{\infty} \frac{n_k^p}{2^{k[p(\gamma+1)+v+1]}}.
\end{aligned}$$

As at the end of Proof of Theorem 2.1, it remains to show that the infinite product Π_t converges for the considered values of t. This is done in a similar way, and we omit the proof. □

Proof of Theorem 2.3. Without loss of generality, we assume that $f(0) = 1$, $f(z_k) = 0$ $(k = 1, 2, \ldots)$. Then by Jensen's inequality

$$\begin{aligned}
J &= \sup_{C_1<R<1} \left(\int_{R/3}^R \left[\int_{C^*}^\tau \frac{n(u)}{u} du \right]^p (1-\tau)^\alpha d\tau \right) (1-R)^\beta \\
&\le C \sup_{C_1<R<1} \int_0^R \left[\int_{\mathrm{T}} \log^+ |f(\tau\xi)| d\xi \right]^p (1-\tau)^\alpha d\tau (1-R)^\beta,
\end{aligned}$$

where $C_1 > 0$ is some constant and $C^* = \tau - (R-\tau)/2$. Estimating the left-hand side of the above inequality from below, we get

$$\begin{aligned}
J &\ge \sup_{C_1<R<1} \left(\int_{R/3}^R \left[n\left(\frac{3\tau - R}{2} \right) \right]^p \left(\frac{R-\tau}{2} \right)^{p+\alpha} d\tau \right) (1-R)^\beta \\
&\ge \sup_{C_1<R<1} \left(\int_{R-\frac{1-R}{3}}^R [n(\rho)]^p (R-\rho)^{\alpha+p} d\rho \right) (1-R)^\beta \\
&\ge \sup_{C_1<R<1} \left(\left[n\left(\frac{3R-1}{2} \right) \right]^p (1-R)^{1+p+\alpha+\beta} \right) \\
&\ge C[n(\rho)]^p (1-\rho)^{1+p+\alpha+\beta},
\end{aligned}$$

where $\rho = (3R-1)/2$ and

$$n(\rho) \le C(1-\rho)^{-(\alpha+\beta+1+p)/p}$$

for any $\rho \in (0,1)$, $\alpha \ge 0$ and $\beta > 0$.

For proving the converse statement, we use Proposition A, Lemma A and the latter inequality for $n(\rho)$. We start by fixing some t for which Lemma A and Proposition A are applicable. Then, similar to Proof of Theorem 2.1,

$$\begin{aligned}
\left\|\Pi_t(z,z_k)\right\|^p_{N^{\infty,p}_{\alpha,\beta}} &\le C \sup_{C_1<R<1}(1-R)^\beta \int_0^R (1-\rho)^\alpha \left[\int_0^1 \frac{(1-t^2)^{t+1}}{(1-\rho t)^{t+1}} n(t)dt\right]^p d\rho \\
&\le C \sup_{C_1<R<1}(1-R)^\beta \int_0^R (1-\rho)^\alpha \left[\int_0^1 \frac{(1-s^2)^{t+1-(\alpha+p+\beta+1)/p}}{(1-\rho s)^{t+1}} ds\right]^p d\rho \\
&\le C \sup_{C_1<R<1} \frac{(1-R)^\beta}{(1-R)^\beta} \le C.
\end{aligned}$$

Now, integrating by parts for $t > (\alpha+\beta+1)/p - 1$ and $\widetilde{\beta} = t - (\alpha+\beta+1)/p > -1$ we get

$$\sum_{|z_k|<R} (1-|z_k|)^{t+2} = \int_0^R (1-s)^{t+2} dn(s) \le C \int_0^R (1-s)^{\widetilde{\beta}} ds < +\infty.$$

Thus, these values of t provide the applicability of Lemma A and Proposition A, and the proof is complete. □

Remark 2.1. It is not difficult to extend the statements and the forthcoming proofs of Theorems 2.1, 2.2, 2.3 to more general, slowly varying weights $w(1-\tau)$ from the class S (see the works [2, 9, 10, 11] and their references).

Remark 2.2. The analogs of Theorems 2.1, 2.2, 2.3 on zero sets and parametric representations are true for the area Nevanlinna type classes in the upper half-plane $\mathbb{C}_+$, which are the analogs of the analytic classes considered above (see [14]).

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