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SHARP FUNCTION ESTIMATE FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we prove the sharp inequality for multilinear commutator related to Littlewood-Paley operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the multilinear commutator.

1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [1]-[4]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss (see [3]) states that commutator $[b, T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [7]-[9], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp inequality for multilinear commutator related to the Littlewood-Paley operator. By using the sharp inequality, we obtain the weighted (L^p, L^q)-norm inequality for the multilinear commutator.

2. Notations and Results

First let us introduce some notations (see [4], [9], [10]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cue Q let $f_Q = |Q|^{-1} \int_Q f(x)dx$

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and the sharp function of f is defined by

$$f^{\#}(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy.$$

It is well-known that (see [4])

$$f^{\#}(x)\approx\sup_{Q\ni x}\inf_{c\in C}\frac{1}{|Q|}\int_Q|f(y)-C|dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $||b||_{BMO}=||b^{\#}||_{L^\infty}$. It has been known that (see [10])

$$||b-b_{2^kQ}||_{BMO}\le Ck||b||_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator, that is that

$$M(f)(x)=\sup_{x\in Q}|Q|^{-1}\int_Q|f(y)|dy;$$

we write that $M_p(f)=(M(|f|^p))^{1/p}$ for $0<p<\infty$. Let $0<\delta<n$, $0<r<\infty$, set

$$M_{r,\delta}(f)(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|^{1-\delta r/n}}\int_Q|f(y)|^rdy\right)^{1/r}.$$

If $0<r\le p<n/\delta$, $1/q=1/p-\delta/n$, we know $M_{r,\delta}$ is type of (p,q), that is

$$\|M_{r,\delta}(f)\|_q\le C\|f\|_p.$$

For $b_j\in BMO(R^n)(j=1,\dots,m)$, set

$$||\vec{b}||_{BMO}=\prod_{j=1}^m||b_j||_{BMO}.$$

Given a positive integer m and $1\le j\le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\dots,\sigma(j)\}$ of $\{1,\dots,m\}$ of j different elements. For $\sigma\in C_j^m$, set $\sigma^c=\{1,\dots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\dots,b_m)$ and $\sigma=\{\sigma(1),\dots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\dots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\dots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO}=||b_{\sigma(1)}||_{BMO}\dots||b_{\sigma(j)}||_{BMO}$.

In this paper, we will study some multilinear commutators as following.

Definition 2.1. Suppose b_j $(j=1,\dots,m)$ are the fixed locally integrable functions on R^n. Let $0<\delta<n$, $\varepsilon>0$ and ψ be a fixed function which satisfies the following properties:

(1) $\int_{R^n}\psi(x)dx=0$,
(2) $|\psi(x)|\le C(1+|x|)^{-(n+1-\delta)}$,
(3) $|\psi(x+y)-\psi(x)|\le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y|<|x|$.

The Littlewood-Paley multilinear commutator is defined by

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] \psi_t(x-y) f(y) dy$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = \int_{R^n} \psi_t(x-y)f(y)dy$, we also define that

$$g_{\psi,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which is the Littlewood-Paley g function (see [11]).

Let H be the space $H = \{h : ||h|| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2}\}$, then, for each fixed $x \in R^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi,\delta}(f)(x) = ||F_t(f)(x)||$$

and

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \ldots = b_m$, $g_{\psi,\delta}^{\vec{b}}$ is just the m order commutator (see [1],[6]). In [5], the sharp estimates for the multilinear commutator $g_\mu^{\vec{b}}$ of another Littlewood-Paley operator g_μ are obtained. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1]-[3], [6]-[9]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our theorems as following.

Theorem 2.1. *Let $b_j \in BMO(R^n)$ for $j = 1, \ldots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(g_{\psi,\delta}^{\vec{b}}(f))^\#(x) \le C \left(M_{r,\delta}(f)(x) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_r(g_{\psi,\delta}^{\vec{b}_{\sigma^c}}(f))(x) \right).$$

Theorem 2.2. *Let $b_j \in BMO(R^n)$ for $j = 1, \ldots, m$. Then $g_{\psi,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$, where $1 < p < n/\delta, 1/q = 1/p - \delta/n$.*

3. Proofs of Theorems

To prove the theorems, we need the following lemma.

Lemma 3.1 (see [11]). *Let $0 < \delta < n$, $1 < p < n/\delta$, $1/q = 1/p - \delta/n$. Then $g_{\psi,\delta}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Lemma 3.2. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \ldots, k$. Then*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le C \prod_{j=1}^k ||b_j||_{BMO}.$$

Proof. Choose $1 < p_j < \infty$, $j = 1, \ldots, m$ such that $1/p_1 + \ldots + 1/p_m = 1$, we obtain, by the Hölder's inequality,

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le \prod_{j=1}^k \left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j} dy\right)^{1/p_j} \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le \prod_{j=1}^k \left(\frac{1}{|Q|}\int_Q |b_j(y) - (b_j)_Q|^{p_j r} dy\right)^{1/p_j r}$$
$$\le \quad C \prod_{j=1}^k ||b_j||_{BMO}.$$

The lemma follows. □

Proof of Theorem 2.1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |g_{\psi,\delta}^{\vec{b}}(f)(x) - C_0| dx \le C \left(||b||_{BMO} M_{r,\delta}(f)(\tilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_r(g_{\psi,\delta}^{\vec{b}_{\sigma^c}}(f)(\tilde{x}))\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$.

We first consider the **Case** $m = 1$**.** Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_{2Q})F_t(f)(x) - F_t((b_1 - (b_1)_{2Q})f_1)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x).$$

Then,

$$\begin{aligned}
& |g_{\psi,\delta}^{b_1}(f)(x) - g_{\psi,\delta}(((b_1)_{2Q} - b_1)f_2)(x_0)| \\
= & \left| ||F_t^{b_1}(f)(x)|| - ||F_t(((b_1)_{2Q} - b_1)f_2)(x_0)|| \right| \\
\leq & ||F_t^{b_1}(f)(x) - F_t(((b_1)_{2Q} - b_1)f_2)(x_0)|| \\
\leq & ||(b_1(x) - (b_1)_{2Q})F_t(f)(x)|| + ||F_t((b_1 - (b_1)_{2Q})f_1)(x)|| \\
& + ||F_t((b_1 - (b_1)_{2Q})f_2)(x) - F_t((b_1 - (b_1)_{2Q})f_2)(x_0)|| \\
= & A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, by the Hölder's inequality with exponent $1/r + 1/r' = 1$, we get

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q A(x)dx \\
= & \frac{1}{|Q|}\int_Q |b_1(x) - (b_1)_{2Q}||g_{\psi,\delta}(f)(x)|dx \\
\leq & C\left(\frac{1}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_{2Q}|^{r'}dx\right)^{1/r'} \left(\frac{1}{|Q|}\int_Q |g_{\psi,\delta}(f)(x)|^r dx\right)^{1/r} \\
\leq & C||b_1||_{BMO}M_r(g_{\psi,\delta}(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose p such that $1 < r < p < q < n/\delta$, $1/q = 1/p - \delta/n$, $r = ps$, by the boundedness of $g_{\psi,\delta}$ on $L^p(R^n)$to $L^q(R^n)$ and the Hölder's inequality, we obtain

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q B(x)dx = \frac{1}{|Q|}\int_Q [g_{\psi,\delta}((b_1 - (b_1)_{2Q})f_1)(x)]dx \\
\leq & \left(\frac{1}{|Q|}\int_{R^n} [g_{\psi,\delta}((b_1 - (b_1)_{2Q})f\chi_{2Q})(x)]^q dx\right)^{1/q} \\
\leq & C\frac{1}{|Q|^q}\left(\int_{R^n} |b_1(x) - (b_1)_{2Q}|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p} \\
\leq & C|Q|^{-1/q+1/ps'+(1-\delta ps/n)/ps}\left(\frac{1}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_{2Q}|^{ps'}dx\right)^{1/ps'} \\
\times & \left(\frac{1}{|2Q|^{1-\delta ps/n}}\int_{2Q} |f(x)|^{ps}dx\right)^{1/ps} \\
= & C|Q|^{-1/q+1/ps'+(1-\delta r/n)/r}\left(\frac{1}{|2Q|}\int_{2Q} |b_1(x) - (b_1)_{2Q}|^{ps'}dx\right)^{1/ps'} \\
\times & \left(\frac{1}{|2Q|^{1-\delta r/n}}|f(x)|^r dx\right)^{1/r} \\
\leq & C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by the Minkowski's inequality, we obtain

$$\begin{aligned}
C(x) &= ||F_t((b_1-(b_1)_{2Q})f_2)(x)-F_t((b_1-(b_1)_{2Q})f_2)(x_0)|| \\
&= \left[\int_0^\infty \left(\int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)||\psi_t(x-y)-\psi_t(x_0-y)|dy\right)^2 \frac{dt}{t}\right]^{1/2} \\
&= \int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)| \left(\int_0^\infty \frac{1}{t}|\psi_t(x-y)-\psi_t(x_0-y)|^2 dt\right)^{1/2} dy \\
&\leq C\int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)| \left(\int_0^\infty \frac{|x_0-x|^{2\varepsilon}\cdot tdt}{(t+|x_0-y|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2} dy \\
&\leq C\int_{(2Q)^c} |b_1(y)-(b_1)_{2Q}||f(y)|\frac{|x_0-x|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta}}dy \\
&\leq C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^kQ} |b_1(y)-(b_1)_{2Q}||f(y)|\frac{|x_0-x|^{\varepsilon}}{|x_0-y|^{n+\varepsilon-\delta}}dy \\
&\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r} \\
&\times \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}|^{r'}dy\right)^{r'} \\
&\leq C\sum_{k=1}^\infty k2^{-k\varepsilon}||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x}) \\
&\leq C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x}),
\end{aligned}$$

thus

$$\frac{1}{|Q|}\int_Q C(x)dx \leq C||b_1||_{BMO}M_{r,\delta}(f)(\tilde{x}).$$

Now, we consider the **Case** $m \geq 2$, we have known that, for $b=(b_1,\dots,b_m)$,

$$\begin{aligned}
& F_t^{\vec{b}}(f)(x) \\
&= \int_{R^n}\left[\prod_{j=1}^m (b_j(x)-b_j(y))\right]\psi_t(x-y)f(y)dy \\
&= \int_{R^n}\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})\psi_t(x-y)f(y)dy \\
&= \sum_{j=0}^m \sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma \int_{R^n}(b(y)-(b)_{2Q})_\sigma \psi_t(x-y)f(y)dy
\end{aligned}$$

$$\begin{aligned}
&= (b_1(x)-(b_1)_{2Q})\dots(b_m(x)-(b_m)_{2Q})F_t(f)(x)\\
&+ (-1)^m F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f)(x)\\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-b(x))_{\sigma^c}\psi_t(x-y)f(y)dy\\
&= (b_1(x)-(b_1)_{2Q})\dots(b_m(x)-(b_m)_{2Q})F_t(f)(x)\\
&+ (-1)^m F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f)(x)\\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}c_{m,j}(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\quad |g^{\vec{b}}_{\psi,\delta}(f)(x)-g_{\psi,\delta}(((b_1)_{2Q}-b_1)\dots((b_m)_{2Q}-b_m))f_2)(x_0)|\\
&= \Big|||F_t^{\vec{b}}(f)(x)||-||F_t(((b_1)_{2Q}-b_1)\dots((b_m)2Q-b_m)f_2)(x_0)||\Big|\\
&\le ||F_t^{\vec{b}}(f)(x)-F_t(((b_1)_{2Q}-b_1)\dots((b_m)_{2Q}-b_m)f_2)(x_0)||\\
&\le ||(b_1(x)-(b_1)_{2Q})\dots(b_m(x)-(b_m)_{2Q})F_t(f)(x)||\\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(\vec{b}(x)-(b_m)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||\\
&+ ||F_t((b_1-(b_1)_{2Q})\dots(b_m-(b_m)_{2Q})f_1)(x)||\\
&+ ||F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(x)-F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_2)(x_0)||\\
&= I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}$$

For $I_1(x)$, by the Hölder's inequality with exponent $1/p_1+\dots+1/p_m+1/r=1$, where $1<p_j<\infty$, $j=1,\dots,m$, we get

$$\begin{aligned}
&\quad \frac{1}{|Q|}\int_Q I_1(x)dx\\
&\le \frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_{2Q}|\dots|b_m(x)-(b_m)_{2Q}||g_{\psi,\delta}(f)(x)|dx\\
&\le \left(\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_{2Q}|^{p_1}\right)^{1/p_1}\dots\left(\frac{1}{|Q|}\int_Q|b_m(x)-(b_m)_{2Q}|^{p_m}dx\right)^{1/p_m}\\
&\quad\times\left(\frac{1}{|Q|}\int_Q|g_{\psi,\delta}(f)(x)|^r dx\right)^{1/r}\\
&\le C||\vec{b}||_{BMO}M_r(g_{\psi,\delta}(f))(\tilde{x}).
\end{aligned}$$

For $I_2(x)$, by the Minkowski's and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx\\
= &\frac{1}{|Q|}\int_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||dx\\
\le &\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||g_{\psi,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|dx\\
\le &C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|g_{\psi,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r}\\
\le &C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_r(g_{\psi,\delta}^{\vec{b}_{\sigma^c}}(f))(\tilde{x}).
\end{aligned}$$

For $I_3(x)$, choose $1<r<p<q<n/\delta$, $1/q=1/p-\delta/n$, $r=ps$, by the boundedness of $g_{\psi,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$, and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx\\
= &\frac{1}{|Q|}\int_Q ||F_t(\prod_{j=1}^m(b_j-(b_j)_{2Q})f_1)(x)||dx\\
\le &\left(\frac{1}{|Q|}\int_{R^n}|g_{\psi,\delta}(\prod_{j=1}^m(b_j-(b_j)_{2Q})f\chi_{2Q})(x)|^q dx\right)^{1/q}\\
\le &C\frac{1}{|Q|^{1/q}}\left(\int_{R^n}|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p}\\
\le &C\frac{1}{|Q|^{1/q}}\left(\int_{2Q}|\prod_{j=1}^m(b_j(x)-(b_j)_{2Q})|^{ps'}dx\right)^{1/ps'}\left(\int_{2Q}|f(x)|^{ps}dx\right)^{1/ps}\\
\le &C|Q|^{-1/q+1/ps'-(1-(\delta ps/n)/ps)}\left(\frac{1}{|2Q|}\int_{2Q}|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})|^{ps'}dx\right)^{1/ps'}\\
&\times\left(\frac{1}{|2Q|^{1-\delta ps/n}}\int_{2Q}|f(x)|^{ps}dx\right)^{1/ps}\\
\le &C||\vec{b}||_{BMO}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, choose $1<p_j<\infty$, $j=1,\dots,m$ such that $1/p_1+\ldots+1/p_m+1/r=1$, we obtain, by the Hölder's inequality,

$$
\begin{aligned}
& I_4(x) \\
= \ & ||F_t(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_2)(x) - F_t(\prod_{j=1}^m (b_j-(b_j)_{2Q})f_2)(x_0)|| \\
= \ & \left(\int_0^\infty \left|\int_{R^n}\left[\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right] f\chi_{(2Q)^c}(y)(\psi_t(x-y)-\psi_t(x_0-y))dy\right|^2 \frac{dt}{t}\right)^{1/2} \\
\leq \ & C\int_{R^n} |\prod_{j=1}^m (b_j(y)-b(b_j)_{2Q})||f\chi_{(2Q)^c}(y)| \left(\int_0^\infty \frac{|\psi_t(x-y)-\psi_t(x_0-y)|^2 dt}{t} dt\right)^{1/2} dy \\
\leq \ & C\int_{(2Q)^c} |\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f(y)| \left(\int_0^\infty \frac{|x-x_0|^{2\varepsilon} t dt}{(t+|x_0-y|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2} dy \\
\leq \ & C\int_{(2Q)^c} |\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f(y)| \frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{(n+\varepsilon-\delta)}} dy \\
\leq \ & C\sum_{k=1}^\infty \int_{2^{k+1}Q\setminus 2^k Q} |x-x_0|^{\varepsilon}|x_0-y|^{-(n+\varepsilon-\delta)}|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f_2(y)|dy \\
\leq \ & C\sum_{k=1}^\infty 2^{-\delta-k\varepsilon}|2^{k+1}Q|^{-1+r\delta/n}\int_{2^{k+1}Q} |\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})||f_2(y)|dy \\
\leq \ & C\sum_{k=1}^\infty 2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}Q|^{1-\delta r/n}}\int_{2^{k+1}Q}|f(y)|^r dy\right)^{1/r} \\
\times \ & \prod_{j=1}^m \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_j(y)-(b_1)_{2Q}|^{p_j}dy\right)^{1/p_j} \\
\leq \ & C\sum_{k=1}^\infty k^m 2^{-km}\prod_{j=1}^m ||b_j||_{BMO} M_{r,\delta}(f)(\tilde{x}) \\
\leq \ & C||\vec{b}||_{BMO} M_{r,\delta}(f)(\tilde{x}),
\end{aligned}
$$

thus

$$\frac{1}{|Q|}\int_Q I_4(x)dx \ = \ C||\vec{b}||_{BMO} M_{r,\delta}(f)(\tilde{x}).$$

This completes the proof of the theorem. □

Proof of Theorem 2.2. We first consider the case m=1, we have

$$\begin{aligned} ||g^{b_1}_{\psi,\delta}(f)||_{L^q} &\leq \|M(g^{b_1}_{\psi,\delta})(f)\|_{L^q} \leq C\|(g^{b_1}_{\psi,\delta}(f))^{\#}\|_{L^q} \\ &\leq C\|M_r(g_{\psi,\delta}(f))\|_{L^q} + C\|M_{r,\delta}(f)\|_{L^q} \\ &\leq C\|g_{\psi,\delta}(f)\|_{L^q} + C\|M_{r,\delta}(f)\|_{L^q} \\ &\leq C\|f\|_{L^p} + C\|f\|_{L^p} \\ &\leq C\|f\|_{L^p}. \end{aligned}$$

When $m \geq 2$, we may get the conclusion of the theorem by induction. This finishes the proof. □

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