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RADIAL DIGRAPHS

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Abstract. The Radial graph of a graph G, denoted by $R(G)$, has the same vertex set as G with an edge joining vertices u and v if $d(u, v)$ is equal to the radius of G. This definition is extended to a digraph D where the arc (u, v) is included in $R(D)$ if $d(u, v)$ is the radius of D. A digraph D is called a Radial digraph if $R(H) = D$ for some digraph H. In this paper, we shown that if D is a radial digraph of type 2 then D is the radial digraph of itself or the radial digraph of its complement. This generalizes a known characterization for radial graphs and provides an improved proof. Also, we characterize self complementary self radial digraphs.

1. Introduction

A directed graph or digraph D consists of a finite nonempty set $V(D)$ of objects called vertices and a set $E(D)$ of ordered pairs of vertices called arcs. If (u, v) is an arc, it is said that u is adjacent to v and also that v is adjacent from u. The outdegree *od* v of a vertex v of a digraph D is the number of vertices of D that are adjacent from v. The indegree *id* v of a vertex v of a digraph D is the number of vertices of D that are adjacent to v. The set of vertices which are adjacent from [to] a given vertex v is denoted by $N_D^+(v)$ $[N_D^-(v)]$. A Digraph D is symmetric if whenever uv is an arc, vu is also an arc. As in Chartrand and Oellermann [7], we use D^* to denote the symmetric digraph whose underlying graph is D. Thus, D^* is the digraph that is obtained from D by replacing each edge of D by a symmetric pair of arcs. For other graph theoretic notations and terminology, we follow [3] and [5].

For a pair u, v of vertices in a strong digraph D the distance $d(u, v)$ is the length of a shortest directed $u - v$ path. We can extend this definition to all digraphs D

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by defining $d(u, v) = \infty$ if there is no directed $u - v$ path in D. The eccentricity of a vertex u, denoted by $e(u)$, is the maximum distance from u to any vertex in D. The radius of D, rad(D), is the minimum eccentricity of the vertices in D; the diameter, diam(D), is the maximium eccentricity of the vertices in D. For a digraph D, the Radial digraph $R(D)$ of D is the digraph with $V(R(D)) = V(D)$ and $E(R(D)) = \{(u, v)/u, v \in V(D) \text{ and } d_D(u, v) = \text{rad}(D)\}$. A digraph D is called a Radial digraph if $R(H) = D$ for some digraph H. If there exist a digraph H with finite radius and infinite diameter, such that $R(H) = D$, then the digraph D is said to be a Radial digraph of type 1. Otherwise, D is said to be a Radial digraph of type 2. For the purpose of this paper, a graph is a symmetric digraph; that is, a digraph D such that $(u, v) \in E(D)$ implies $(v, u) \in E(D)$. Our first result gives a useful property of radial digraphs. The proof is straightforward, so we omit it.

Lemma 1.1. *If D is a symmetric digraph, then $R(D)$ is also symmetric.*

The converse of Lemma 1.1 need not be true. Figure 1 shows an asymmetric strong digraph D of order $p = 4$ with rad$(D) = 2$ and diam$(D) = 3$ and the corresponding symmetric radial digraph $R(D)$.

FIGURE 1.

The convention of representing the symmetric pair of arcs (u, v) and (v, u) by the single edge uv induces a one-to-one correspondence ϕ from the set of symmetric digraphs to the set of graphs. For example in Figure 1, we have $\phi(R(D)) = K_2 \bigcup K_2$. Therefore, by Lemma 1.1, it is natural to define, for a graph G, the Radial graph $R(G)$ of G as the graph with $V(R(G)) = V(G)$ and $E(R(G)) = \{uv/u, v \in V(G) \text{ and } d_G(u, v) = \text{rad}(G)\}$.

The concept of antipodal graph was initially introduced by [13]. The antipodal graph of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph H.

Aravamudhan and Rajendran [1] and [2] gave the characterization of antipodal graphs. After that, Johns [8] gave a simple proof for the characterization of antipodal graphs. Motivated by the above concepts, Kathiresan and Marimuthu [9], [10], [11] and [12] introduced a new type of graph called Radial graphs and the following properties of radial graphs have been verified.

Proposition 1.1. [12] *If* $\mathrm{rad}(G) > 1$*, then* $R(G) \subseteq \overline{G}$*.*

Theorem 1.1. [12] *Let* G *be a graph of order* n*. Then* $R(G) = G$ *if and only if* $\mathrm{rad}(G) = 1$*.*

Let $S_i(G)$ be the subset of the vertex set of G consisting of vertices with eccentricity i.

Lemma 1.2. [12] *Let* G *be a graph of order* n*. Then* $R(G) = \overline{G}$ *if and only if* $S_2(G) = V(G)$ *or* G *is disconnected in which each component is complete.*

Theorem 1.2. [12] *A graph* G *is a radial graph if and only if it is the radial graph of itself or the radial graph of its complement.*

2. A Characterization of Radial Graphs using Radial Digraphs

We begin with some properties of Radial digraphs.

Lemma 2.1. *If* $\mathrm{rad}(D) > 1$*, then* $R(D) \subseteq \overline{D}$*.*

Proof. By the definition of $R(D)$ and $\overline{D}$, we have $V(R(D)) = V(\overline{D}) = V(D)$. If (u, v) is an arc of $R(D)$, then $d_D(u, v) = \mathrm{rad}(D) > 1$ in D and hence $uv \notin E(D)$. Therefore, $uv \in E(\overline{D})$. Thus, $E(R(D)) \subseteq E(\overline{D})$. Hence $R(D) \subseteq \overline{D}$. □

As a special case, we have Proposition 1.1.

Theorem 2.1. *Let* D *be a digraph of order* p*. Then* $R(D) = D$ *if and only if* $\mathrm{rad}(D) = 1$*.*

Proof. Let D be a digraph of order p. Suppose $\mathrm{rad}(D) = 1$. Then, by the definition of radial digraph, $R(D) = D$.

Conversely, assume that $R(D) = D$. Suppose $\mathrm{rad}(D) \neq 1$. Then, by Lemma 2.1 we have $R(D) \subseteq \overline{D}$, a contradiction. Hence, $\mathrm{rad}(D) = 1$. □

If D is a symmetric digraph of radius 1, then $\phi(D)$ is a graph of radius 1. This implies Theorem 1.1.

We now present a result that will be useful in our characterization of radial digraphs. Let $S_i(D)$ be the subset of the vertex set of D consisting of vertices with eccentricity i.

Theorem 2.2. *Let D be a digraph of order p. Then $R(D) = \overline{D}$ if and only if any one of the following holds.*

(a) $S_2(D) = V(D)$,

(b) *D is not strongly connected such that for any $v \in V(D)$, od $v < p-1$ and for every pair u, v of vertices of D, the distance $d_D(u, v) = 1$ or $d_D(u, v) = \infty$.*

Proof. If $S_2(D) = V(D)$, then $(u, v) \in E(R(D))$ if and only if $(u, v) \notin E(D)$. Also, there are no vertices u and v in D such that $d_D(u, v) > 2$. Hence, $R(D) = \overline{D}$. Now, suppose that b holds. If $d_D(u, v) = \infty$ for every pair u, v of vertices, then $D = \overline{K_p^*}$ for some positive integer p and $R(D) = R(\overline{K_p^*}) = K_p^* = \overline{D}$. Since for any $v \in V(D)$, *od* $v < p-1$, $\mathrm{rad}(D) \neq 1$. Hence $\mathrm{rad}(D) = \infty$. In this case, if $(u, v) \in E(D)$, then $(u, v) \notin E(R(D))$. If $(u, v) \notin E(D)$, then $d_D(u, v) = \infty$ and so $(u, v) \in E(R(D))$. Hence, $R(D) = \overline{D}$.

Conversely, assume that $R(D) = \overline{D}$.

Case 1. Suppose that the radius is finite. Assume $\mathrm{rad}(D) \neq 2$. If $\mathrm{rad}(D) = 1$, then by Theorem 2.1, $R(D) = D$, which is a contradiction to $R(D) = \overline{D}$. Thus, we assume that $2 < \mathrm{rad}(D) < \infty$. Let u and v be vertices of D such that $d_D(u, v) = 2$. Note that $(u, v) \notin E(D)$ and $(u, v) \notin E(R(D))$; so $R(D) \neq \overline{D}$, which is a contradiction.

Case 2. Assume that the radius is infinite. Then there exist vertices u and v such that $1 < d_D(u, v) < \infty$. Then $(u, v) \notin E(D)$ and $(u, v) \notin E(R(D))$ and again $R(D) \neq \overline{D}$, which is a contradiction. Hence, $\mathrm{rad}(D) = 2$.

There are two possibilities $\mathrm{rad}(D) = \mathrm{diam}(D) = 2$ and $\mathrm{rad}(D) = 2$, $\mathrm{diam}(D) > 2$. It is well known that $\mathrm{rad}(D) \leq \mathrm{diam}(D)$. Suppose that $\mathrm{rad}(D) < \mathrm{diam}(D)$. Let x and y be vertices in D such that $d_D(x, y) = \mathrm{diam}(D)$. Now $(x, y) \notin E(D)$ implies $(x, y) \in E(\overline{D})$. But $(x, y) \notin E(R(D))$, a contradiction to $R(D) = \overline{D}$. Hence, the only possibility is $\mathrm{rad}(D) = \mathrm{diam}(D) = 2$. □

If D is a self centered symmetric digraph of radius 2, then $\phi(D)$ is a self centered graph of radius 2. On the otherhand, if D is symmetric but not strongly connected such that for any $v \in V(D)$, $od\ v < p-1$ and for every pair u and v of vertices of D, the distance $d_D(u,v) = 1$ or $d_D(u,v) = \infty$, then $\phi(D)$ is a disconnected graph where each component is complete. This implies Lemma 1.2.

We now give a characterization of radial graphs using radial digraphs of type 2.

Theorem 2.3. *If D is a radial digraph of type 2, then D is the radial digraph of itself or the radial digraph of its complement.*

Proof. Suppose that D is a radial digraph of type 2 and let H be a digraph such that $R(H) = D$. We consider three cases based on H.

Case 1. Suppose that $\text{rad}(H) = 1$. Then by Theorem 2.1, $R(H) = H$.

Case 2. Suppose that $1 < \text{rad}(H) < \infty$. Then the diameter of H may be finite or infinite. Since D is a radial digraph of type 2, diameter of H is finite. Then H is strongly connected and for every pair u, v of vertices of H, the distance $d_H(u,v) < \infty$. Define H' as the digraph formed from H by adding the arc (u,v) to $E(H)$ if $d_H(u,v) \neq \text{rad}(H)$. Note that $d_{H'}(u,v) = 1$ when $d_H(u,v) \neq \text{rad}(H)$ and $d_{H'}(u,v) = 2$ when $d_H(u,v) = \text{rad}(H)$. Hence for every vertex v in H', there exist a vertex which are at distance $\text{rad}(H)$. Thus, $D = R(H) = R(H')$. Since $\text{rad}(H') = \text{diam}(H') = 2$, by Theorem 2.2 we have $R(H') = \overline{H'}$. Therefore, $D = \overline{H'}$ and $\overline{D} = H'$ which gives $D = R(\overline{D})$ as desired.

Case 3. Suppose that $\text{rad}(H) = \infty$. Define H' as the digraph formed from H by adding the arc (u,v) to $E(H)$ if $d_H(u,v) \neq \text{rad}(H)$. Now, if $d_H(u,v) < \infty$, then $d_{H'}(u,v) = 1$ and if $d_H(u,v) = \infty$, then $d_{H'}(u,v) = \infty$. Thus, $D = R(H) = R(H')$. Since for every pair u, v of vertices of H', the distance $d_{H'}(u,v) = 1$ or $d_{H'}(u,v) = \infty$, by Theorem 2.2 we have $R(H') = \overline{H'}$. Therefore, $D = \overline{H'}$ and $\overline{D} = H'$ which gives $D = R(\overline{D})$ as desired. □

If D is a symmetric radial digraph of type 2, then $\phi(D)$ is a radial graph. Also by Theorem 2.3, G is the radial graph of itself or the radial graph of its complement. This implies the characterization of radial graphs in Theorem 1.2 follows immediately.

Figure 2 shows that D is a digraph on four vertices whose radius is one and diameter is infinite such that $R(D) = D$, but D is not a radial digraph of type 2.

FIGURE 2.

Figure 3 shows that D is a radial digraph since there exist only one digraph H on four vertices whose radius is two and diameter is infinite. Also, H is neither D nor $\overline{D}$ and hence D is a radial digraph of type 1. Since there is no relationship between radial digraphs of type 1 and graphs, we have not considered such digraphs in this paper. So, we propose the following problem.

Problem: Characterize Radial digraphs of type 1.

FIGURE 3.

In view of the Theorem 2.3, it is natural to ask whether there exist a digraph which are not radial digraph of type 2. The next theorem answers this question.

Theorem 2.4. *A disconnected digraph D is a radial digraph of type 2 if and only if each vertex in D has outdegree at least one.*

Proof. Let D be a radial digraph of type 2. Then there exist a graph H such that $R(H) = D$ where H is either D or $\overline{D}$. Since D is disconnected, $R(D)$ and $\overline{D}$ are connected. Hence $R(D) \neq D$. Assume that the components of D contains a vertex (say u) whose outdegree is zero. Then by definition of $R(D)$, there is an arc from u to every other vertex in D. Hence $\text{rad}(R(D)) = 1$ and so $R(D) \neq D$. Also $\text{rad}(\overline{D}) = 1$, by Theorem 2.1 we have $R(\overline{D}) = \overline{D}$. Hence $R(\overline{D}) \neq D$. Hence each vertex in D has outdegree at least one.

For the converse, suppose each vertex in D has outdegree at least one. Since for every vertex (say u) in D there is an arc from u to at least one vertex in the same component, $d_{\overline{D}}(u, v) = 1$ if $u \in D_i$, $v \in D_j$, $i \neq j$ and $d_{\overline{D}}(u, v) = 2$ if $u, v \in D_i$.

Hence, $e_{\overline{D}}(u) = 2$, for all $u \in V(\overline{D})$ and so $R(\overline{D})$ is disconnected which is equal to D. Hence, D is a radial digraph. □

If each vertex in symmetric digraph D has outdegree at least one, then $\phi(D)$ is a graph which has no K_1 component. Therefore, we have the following result.

Theorem 2.5. *A disconnected graph G is a radial graph if and only if G has no K_1 component.*

3. Self-Radial Digraphs and Graphs

In the previous section, we proved, for a digraph D of order p, that the radial digraph $R(D)$ is identical to D if and only if $\text{rad}(D) = 1$. Similarly, for a graph G of order p, the radial graph $R(G)$ is identical to G if and only if $\text{rad}(G) = 1$. A more interesting question can also be asked. When is $R(D)$ isomorphic to D or when is $R(G)$ isomorphic to G? If $R(D) \cong D$, then we will call D a self radial digraph and if $R(G) \cong G$, we will call G a self radial graph.

For a class of self radial digraphs D that are strongly connected, given a positive integer $p \geq 3$, the directed cycle C_p' where $V(C_p') = \{v_1, v_2, \ldots, v_p\}$ and $E(C_p') = \{(v_i, v_{i+1})/1 \leq i \leq p-1\} \bigcup \{(v_p, v_1)\}$ is self radial.

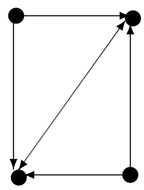

Figure 4.

The self radial digraph D in Figure 4 is an example of minimum order that is weakly connected but not unilaterally connected.

Proposition 3.1. *There exist a family of self radial digraphs which are unilaterally connected but not strongly connected.*

Proof. For a digraph D' on $p \geq 4$ vertices, where $V(D') = \{v_1, v_2, \ldots, v_p\}$ and $E(D') = \{(v_i, v_{i+1})/1 \leq i \leq p-2\} \bigcup \{(v_{p-1}, v_1)\} \bigcup \{(v_1, v_p)\}$. Then $e(v_1) = p-2$, $e(v_2) = p-1$, $e(v_i) = p-2$, $3 \leq i \leq p-1$ and $e(v_p) = \infty$. Thus, $\text{rad}(D') = p-2$ and $\text{diam}(D') = \infty$. We define a mapping f between D' and $R(D')$ as follows: $f(v_1) = 3$,

$f(v_2) = 2$, $f(v_3) = 1$, $f(v_p) = p$ and $f(v_i) = p + 3 - i$, $4 \le i \le p - 1$. The mapping f is an isomorphism between D' and $R(D')$ and so D' is self radial. □

Proposition 3.2. *If D is a disconnected digraph, then D is not self radial.*

Proof. Let u and v be vertices of D. If u and v are in different components of D, then $d_D(u, v) = \infty = \text{rad}(D)$. Thus, $(u, v) \in E(R(D))$ and $d_{R(D)}(u, v) = 1$. If u and v are in the same component of D, then there exists a vertex w in the second component of D. Now, $d_D(u, w) = d_D(w, v) = \infty$, so $(u, w) \in E(R(D))$ and $(w, v) \in E(R(D))$ and $d_{R(D)}(u, v) \le 2$. Therefore, $R(D)$ is strongly connected and $\text{rad}(R(D)) \le 2$. □

Proposition 3.3. *A self centered digraph of radius 2 is self radial if and only if D is self complementary.*

Proof. Let D be a self centered digraph of radius 2. Then $R(D) = \overline{D}$. Since D is self radial, $R(D) \cong D$. Hence, $D \cong \overline{D}$.

Conversely, let D be self complementary. Since D is a self centered digraph of radius 2, $R(D) = \overline{D}$. Hence, $R(D) \cong D$ and so D is self radial. □

Theorem 3.1. *A self complementary digraph D is self radial if and only if D is self centered digraph of radius 2.*

Proof. Since D is self complementary and self radial, $R(D) \cong D$. Suppose D is a self centered digraph with $\text{rad}(D) \ge 3$. Then $\overline{D}$ is a self centered digraph of radius 2. Hence, $\text{rad}(R(D)) \ne \text{rad}(\overline{D})$, which is a contradiction. Hence, D is a self centered digraph of radius 2.

Conversely, if D is a self centered digraph of radius 2, then by Theorem 2.2 we have $R(D) = \overline{D}$. Since D is self complementary, $R(D) \cong D$. Hence, D is self radial. □

Theorem 3.2. *If D is a self radial digraph with $R(D) \ne D$, then $p \le q(D) \le p(p-1)/2$.*

Proof. If D is disconnected, then by Proposition 3.2 we have D is not self radial. The minimum number of arcs in a connected digraph is $p - 1$. Then D can contain no directed cycles and hence D is not strongly connected. If D is unilaterally connected, then D can contain a directed walk that passes through each vertex of D. This can only be done with $p - 1$ arcs if D itself is a directed path $P' : v_1, v_2, \ldots, v_p$. However in $R(P')$, there is only one arc from v_1 to v_p and so $R(P')$ is disconnected. Hence D

is not self radial. Finally, if D is a weakly connected but not unilaterally connected, then the radius may be finite or infinite. If the radius of D is finite and $e(v) = \text{rad}(D)$, then there exist a vertex $u \in N_D^+(v)$ and there is no path from u to v and so $e(u) = \infty$. If we take any vertex $v' \in D$, the directed distance from u to v' (v' to u) is either less than the radius of D or infinity, then in $R(D)$, u must be an isolated vertex. Since D is connected and $R(D)$ contains an isolated vertex, D is not self radial. Suppose the radius of D is infinite, then there exist two vertices u and v in D such that no $u - v$ directed path and no $v - u$ directed path exist in D. Therefore, both the arcs (u, v) and (v, u) are in $R(D)$. Since D contains no directed cycles and $R(D)$ contains a directed 2-cycles, D is not self radial. Hence, $q(D) \geq p$.

For the upperbound, since $R(D) \neq D$, $R(D) \subset \overline{D}$. Now, $D \cong R(D) \subset \overline{D}$ implies that $q(D) \leq q(\overline{D})$. Hence, $q(D) \leq \frac{1}{2}q(K_p^*) = p(p-1)/2$. □

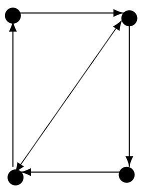

FIGURE 5.

The bounds in Theorem 3.2 are sharp. The lower bound is sharp for the class of directed cycles and Figure 5 is an example of minimum order self complementary self centered digraph of radius 2 which satisfies the sharpness of the upperbound.

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