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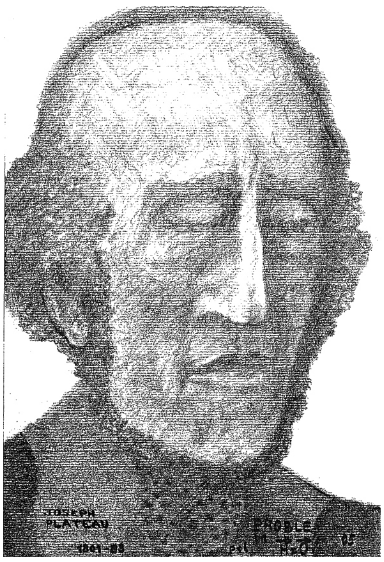
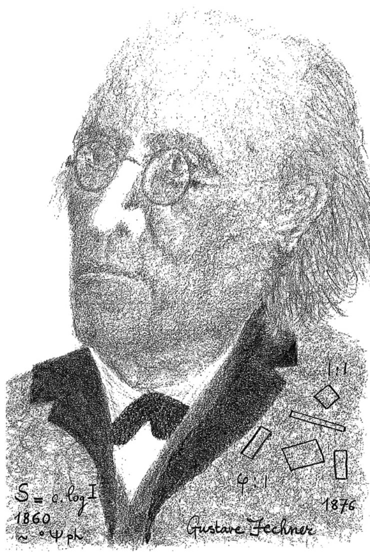
A GEOMETRICAL DESCRIPTION OF VISUAL SENSATION II

A COMPLEMENTED MODEL FOR VISUAL SENSATION EXPLICITLY TAKING INTO ACCOUNT THE LAW OF FECHNER, AND ITS APPLICATION TO PLATEAU'S IRRADIATION

BART ONS [1] AND PAUL VERSTRAELEN [2]





Joseph Plateau and Gustav Fechner

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Abstract. Plateau's irradiation phenomenon in particular describes what one sees when observing a brighter object on a darker background and a physically congruent darker object on a brighter background: the brighter object is seen as being larger. This phenomenon occurs in many optical visual illusions and it involves some fundamental aspects of human vision. We present a general geometrical model of human visual sensation and perception, hereby taking into account the law of Fechner in addition to the anisotropic smoothing that was introduced in [1], and explicitly illustrate its meaning for irradiation illusions of Helmholtz and Kitaoka.

Introduction

In our previous article *"A geometrical description of visual sensation"* [1] the main purpose was to introduce *anisotropic* (i.e. properly elliptical) *Gaussian smoothing* (with kernels $G_0(x, y; a_h, a_v)$, whereby the horizontal axis a_h is smaller than the vertical axis a_v) of *planar light intensities* $\big(I(x, y)\big)$ so as to define *human visual sensation* in a way which basically takes into account human's factually *different appreciations of horizontal and vertical dimensions* (thus considering $I(x, y; a_h, a_v) = I(x, y) \otimes G_0(x, y; a_h, a_v)$ as the observed luminosities for apertures a_h, a_v). Then, knowing that *perception* essentially is determined by *the Casorati curvature* of the corresponding relief surface (that is, of the surface consisting of the points $\big(x, y, z = I(x, y; a_h, a_v)\big)$ in R^3) cfr. a.o. [2] and [3], in [1] we illustrated the relevance of such anisotropic smoothing in the context of the perception of visual stimuli which are tilted elliptical Gaussian blobs, referring to [4] for concrete experimental data in this respect. In [1], we announced our intention to deal later on with more general stimuli as well. When actually setting out to do so we realised that, still aiming merely for the modest goal of very roughly and just qualitatively to scientifically define human visual sensation, it seemed likely to be best right from the start further to add, and this not "between the lines" as was done in [1] but rather explicitly, *the law of Fechner* (i.e. to consider visual sensation $F(x, y) = k \ln I(x, y; a_h, a_v)$ for some constant k); this will be done in Section 1. And, then, in Section 2, (and as by now might demand less of some reader's goodwill in their consideration of our views on the matters of sensation and of perception -at least eventually, since, as Albert Einstein stated: *"The truth of a theory is in your mind, not in your eyes."*-), the "border lines" between 2D regions of distinct luminosities $\big(I(x, y)\big)$ as these are fixed by the extrema of the Casorati surfaces $\big(x, y, z = \mathcal{C}(x, y)\big)$ of the relief surfaces of visual sensation (i.e. of the surfaces consisting of the points $\big(x, y, z = F(x, y)\big)$ in R^3) will be shown in the particular instance of a standard version of the so–called *"brightness illusions"*, namely for the *illusion of Helmholtz* (cfr. [5],[6]). In Section 3 we apply our geometrical model to *Kitaoka's "bulge illusion"* [7], i.e. an irradiation illusion that on a first sight looks more complicated than Helmholtz's illusion and which became kind of popular ever since its introduction in 1998. In Section 4 we will give somewhat more information on the history of the study of the phenomenon of *irradiation* as well as some comments on the very *nature of science* via some quotes of o.a. Plateau, Fechner, Helmholtz, Schrödinger and Minnaert. At last, we thought that it might be not amiss to include two small Appendices, one about *geometry* and one about *psychophysics*, treating some of their aspects in a bit more sophisticated way than was done in the preceding text, but showing at the same time that the more brutal approach that was given before pretty well "does its job" from a practical point of view and is likely so much less demanding to visualise in our minds.

1. A geometrical description of human visual sensation

Along the lines drawn in [1], also at present we merely intend to give a basic, qualitative and direct geometrical description of human visual sensation, (e.g. hereafter we will not care about the inhomogeneity of the human visual field, nor about related Finsler–Riemann anisotropies of the image planes). But, in comparison with the proposal made before, now we will moreover explicitly take into account *Fechner's law* in the definition of sensation, rather than to just mention its relevance in passing, since this is of crucial importance for a better understanding of the definition of human visual perception, in general, and more in particular, for a better understanding of Plateau's irradiation phenomenon.

Thus, let $I(x, y)$ be the *luminosity* of a given *image* in an (x, y)–plane R^2, which will be extended in a z–direction R as a *scalar field* on R^2. Consider, thereby following [1], its *elliptical Gaussian smoothing* $I(x, y; a_h, a_v) = I(x, y) \otimes G_0(x, y; a_h, a_v)$ whereby $G_0(x, y; a_h, a_v) = e^{-[(x/a_h)^2+(y/a_v)^2]/2}/2\pi a_h a_v$, for suitable $a_h < a_v$. Then, in view of the law of Fechner (cfr. [8],[9],[10],[11],[12]), we define the *human visual sensation* corresponding to the given image by $F(x, y) = S(x, y; a_h, a_v) = k \ln I(x, y; a_h, a_v)$, whereby k is some real constant. And, according to e.g. [2],[3], the corresponding *human visual perception* is basically defined via the *Casorati curvatures*, i.e., by the scalar valued extrinsic curvature measures which best reflect the human intuitive numerical curvature appreciations of the relief surface of *the Fechner sensation of the given image*, i.e. of the graph $\big(x, y, z = F(x, y)\big)$ in R^3 of the sensation function F.

In 1D, i.e., for curves in a plane R^2 which are the graphs $\big(x, I(x)\big)$ of some function $I : R \to R$ instead of in 2D, i.e. for surfaces in a space R^3, and by smoothing the signal with a one–dimensional Gaussian kernel with aperture s, the above could be illustrated as follows, cfr. Figure 1.

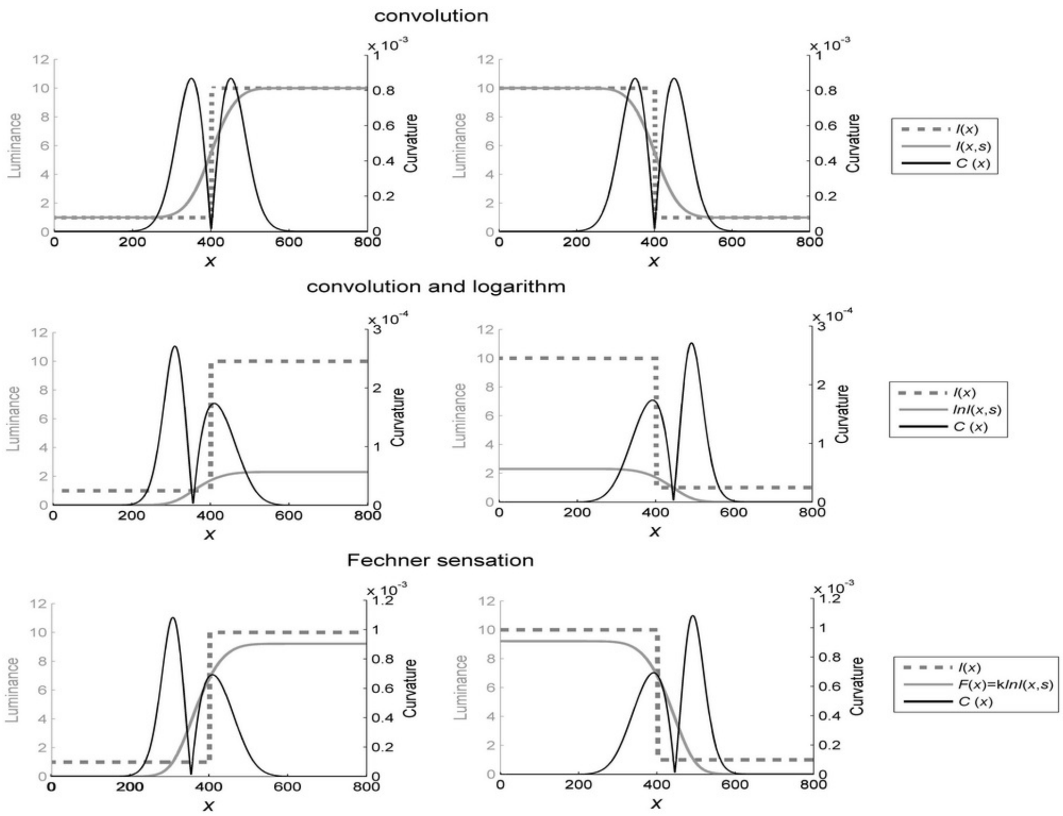

FIGURE 1. Fechner sensation for one–dimensional luminance step functions I and their Euclidean curvatures $\mathcal{C}$.

On the first row of this figure, one sees an "image" consisting of two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ with respective intensities I_1 and I_2, displayed left $I_1 < I_2$ and right for $I_1 > I_2$, having $I(x) = I_1$ on $\mathcal{R}_1 : x \le 400$ and $I(x) = I_2$ on $\mathcal{R}_2 : x > 400$ (the dotted gray line), its Gaussian smoothing $I(x, s)$ with aperture s (the gray line) and the latter's curvature $\mathcal{C}(x) = |I''(x, s)/[1 + I'^2(x, s)]^{\frac{3}{2}}|$, (whereby the accents $'$ and $''$ denote the first and the second derivatives with respect to x; the black line). On the second row, one sees, for the same image intensities $I(x)$ (dotted gray line), their logarithms (gray line) and the curvatures (black line) of these logarithms. Finally, on the last row of Figure 1, still for the same image curves I (dotted gray line), one sees the "Fechner sensations" $F(x) = k \ln I(x,s)$ (gray line), for some constant k, and the curvatures $\mathcal{C}$ (black line) of these planar F curves. In the examples of Figure 1, the transition between the distinct intensities I_1 and I_2, which is located at $x = 400$ in the "ideal" image (cfr. [12]), would "happen" at $x \approx 300$ and at $x \approx 500$) in the transition "from lower to higher" and "from higher to lower", respectively, when looking at the maximum of the curvature and would "happen" at $x \approx 350$ and at $x \approx 450$, respectively, when looking at the minimum of the curvature.

2. Helmholtz's illusion as case study of Plateau's irradiation phaenomenae

The *illusion of Helmholtz*, as depicted in Figure 2, essentially concerns the fact that a white square on a black background looks larger than a physically congruent dark square on a white background; (cfr. e.g. [5], [6], [13], [14]).

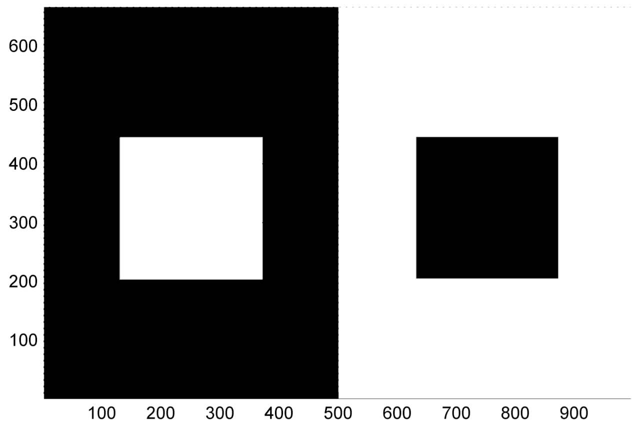

Figure 2. The illusion of Helmholtz, (the numbers on the axes represent distances in some arbitrary unit).

In Figure 3 are then presented: (A), the graphs of the intensities $I_1 < I_2$, whereby $I = I_1(x,y)$ for points (x,y) belonging to the white square and $I = I_2(x,y)$ for points not belonging to the white square in the left part of Figure 2, and, vice–versa, with arbitrary discrete luminance values I_1 and I_2, (which are put equal to 10 and 1 vertical units here, just by way of example); (B), the graphs of their *anisotropic Gaussian* *smoothing*, (the hereby applied values for a_h and a_v are 4 and 6 units, respectively, having a ratio $a_h : a_v$ equal to $2 : 3$, [of course, one may feel very welcome indeed to refine or better calibrate to one's own reality the scales that were used in our figures, (we repeat to envisage here and now only a fundamental point of view on the matters of human visual sensation and perception)]; (C), the Fechner sensations of the two "original" images, i.e. the surfaces $z = F(x, y) = S(x, y; a_h, a_v) = k \ln I(x, y; a_h, a_v)$, (whereby actually $k = 2$ was used, again just by way of example); (D), the *corresponding Casorati surfaces*, i.e. the surfaces $z = \mathcal{C}(x, y) = \frac{1}{2}\|h(x, y)\|^2 = \frac{1}{2}([k_1^2 + k_2^2])(x, y)$, whereby h and k_1 and k_2 respectively denote the *second fundamental form* and *Euler's principal curvatures* of the surfaces $z = F(x, y)$ in the $3D$ Euclidean space at the points of these surfaces lying in R^3 above the points with Cartesian co–ordinates (x, y) in the image plane R^2 [15], [16], [17].

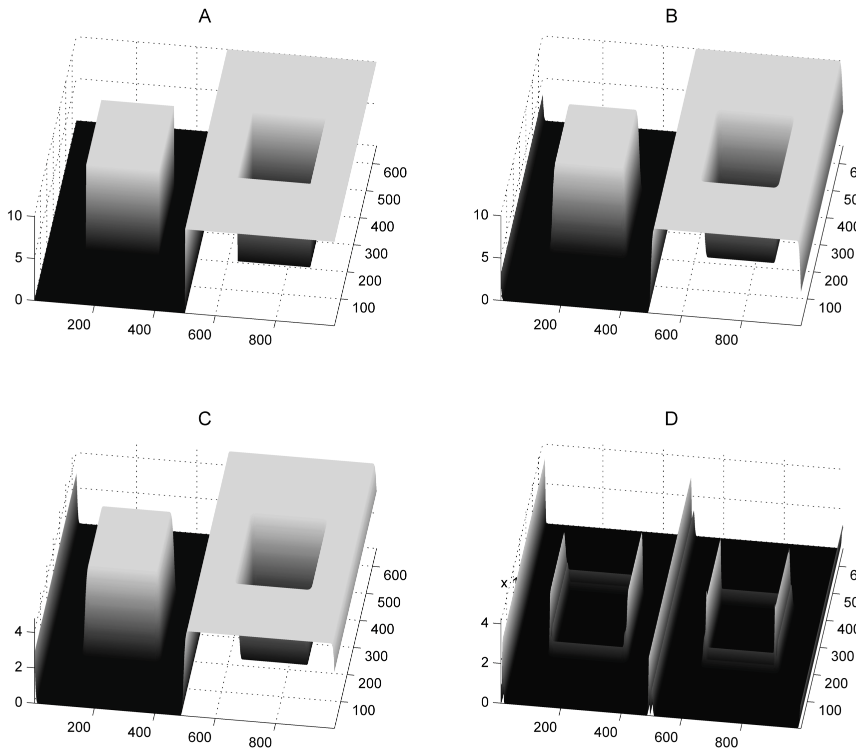

FIGURE 3. The reliefs of the functions $I(x, y)$ (A), $I(x, y; a_h, a_v)$ (B), $F(x, y)$ (C) and $\mathcal{C}(x, y)$ (D) for the illusion of Helmholtz.

Finally, in Figure 4 are shown the border lines between the square regions and their backgrounds which are presented in Figure 2, as these border lines are determined by the extremal values of the Casorati curvatures of the Fechner surfaces $z = F(x, y)$ in the two situations at hand, "white on black" and "black on white", (whereby in full agreement with the previously announced direct and non–subtle present approach, at this stage, we completely neglect an albeit pretty interesting, more delicate analysis of what there is actually to be seen in the very near vicinity of the vertices of these squares, or still, whereby we restrict to our at most unsophisticated "early vision").

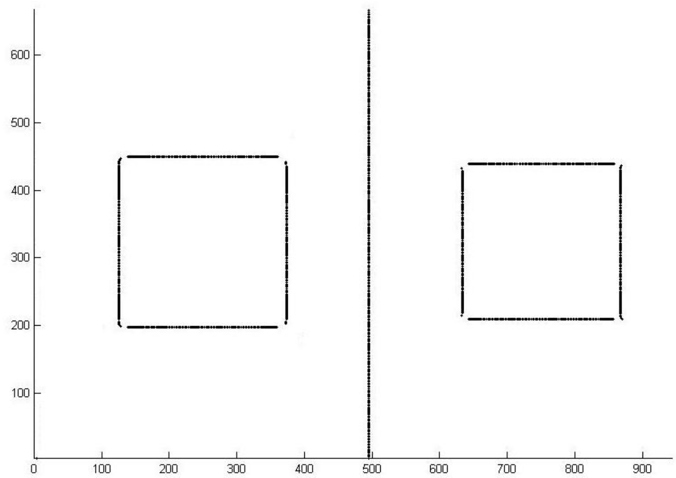

FIGURE 4. "Und deines Geistes höchster Feuerflug
Hat shon am Gleichnis, hat am Bild genug", (Goethe).

Hereby, naturally (in view of the horizontal–vertical effect), *proper rectangles* do result: "a rectangle standing up" in the left case, and a "rectangle lying down" in the right case. This latter observation in itself, i.e., in particular, actually seeing a brighter "horizontal–vertical" square on a darker background as a standing up rectangle, is a (though not so often "treated") manifestation of a brightness–illusion (cfr. e.g. [7], [13]); moreover this further illustrates the relevance of the anisotropic Gaussian smoothing of the images' luminosities as this was introduced in [1]. And, concerning the comparison of the perceptions of the sizes of the white and black squares, which in the original images both have horizontal and vertical sides of 241 units, with the parameters a_h, a_v and k chosen above, the "bases and heights" of the rectangles which are –in the meaning of this word at the time of the "Old Greeks"– *the "phaenomenae"*, i.e. : *the appearances to us of these squares*, are 249 and 253 units for the "white on black" square, respectively, and 233 and 229 units for the "black on white" square, respectively. (In addition, as one more manifestation of irradiation, one may observe that the vertical line separating the two halves of Figure 2 is situated essentially left of the 500 mark in Figure 4.)


Herman Helmholtz and Felice Casorati

3. One further illustration: Kitaoka's bulge illusion

The *bulge illusion*, as depicted in Figure 5, essentially concerns the fact that the alternating black and white main squares of the picture are perceived as more and

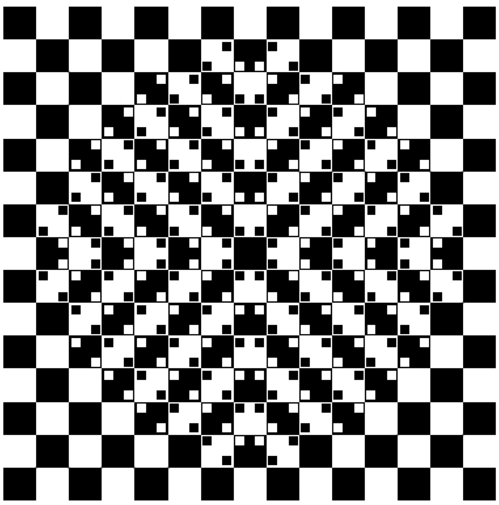

Figure 5. The standard image of Kitaoka's bulge illusion.

more deformed 4–gons when proceeding from the boundaries to the center resulting in curved perceived lines lying in between the rows and columns of the squares, whereas, physically, all seemingly non–squares are mutually congruent real squares indeed, and, consequently, all the lines between these true squares are physically perfectly horizontal and vertical lines.

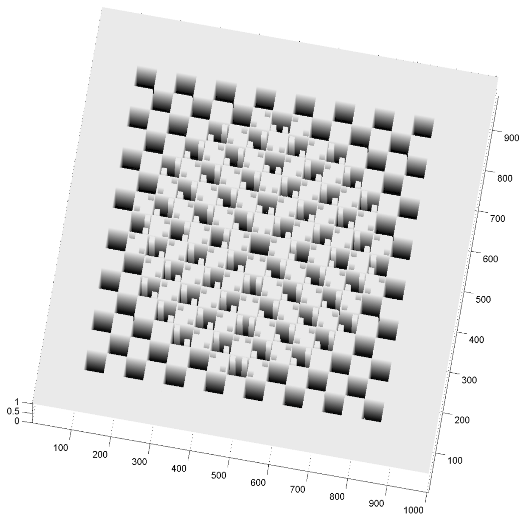

FIGURE 6. A view on the corresponding relief surface $z = I(x, y)$.

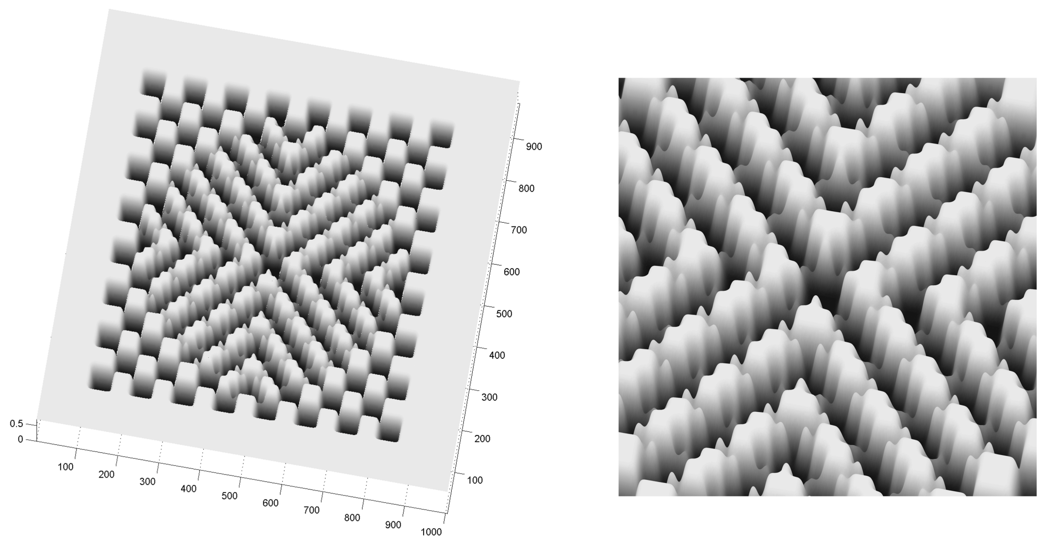

FIGURE 7. The corresponding Fechner sensation surface $z = F(x, y)$ and a central close–up.

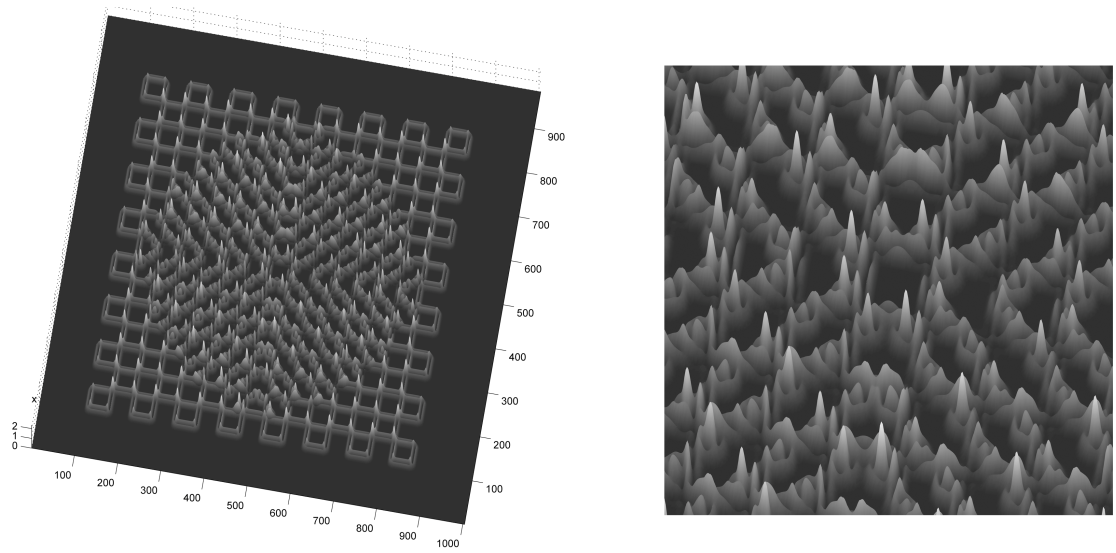

FIGURE 8. The Casorati surface $z = \mathcal{C}(x, y)$ and a central close–up.

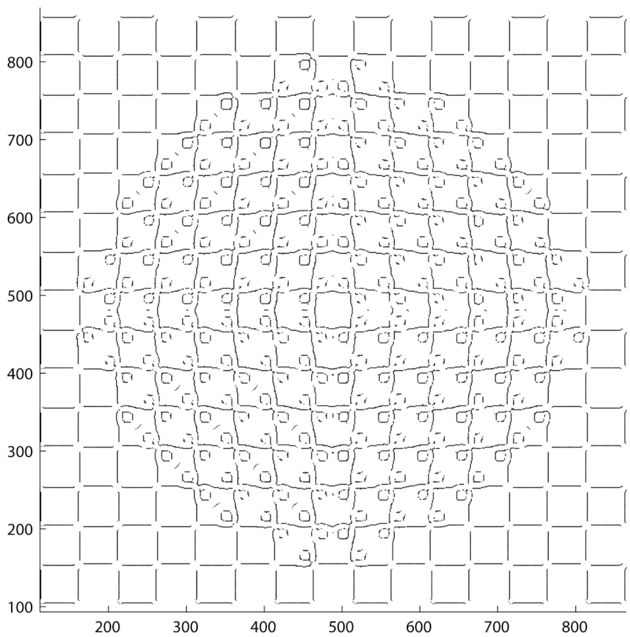

FIGURE 9. The Casorati surface's main extrema.

4. Some more or less appropriate further citations and comments

From Maurice Dorikens' book *"Joseph Plateau (1801–1883). Leven tussen Kunst en Wetenschap, Vivre entre l'Art et la Science, Living between Art and Science"*, (Provincie Oost–Vlaanderen, België, 2001), we learned the following. Plateau presented a detailed description and an analysis of his irradiation experiments to the Belgian Academy in November 1838 and this presentation was published in 1839 as *"Mémoire sur l'irradiation"*. This article also contains a bibliography concerning irradiation from Epicurus onwards. *"Irradiation" describes what one sees when observing a brighter object on a darker background:* the image formed on the retina "s'étend un peu au–delà de l'espace directement excité", as Plateau described it, i.e. : *the brighter object is seen as being larger than it actually is.* As an example, Plateau a.o. treated the diameter of the moon, as follows (see Figure 10). In the first quarter of the moon, when it forms a crescent, one sees the lit crescent extend beyond the remaining dark disc of the moon, and thus seem larger. To systematically study this effect, Plateau constructed a suitable instrument and, based on the measurements made with it, deduced a number of properties of irradiation on which the quoted article reported.

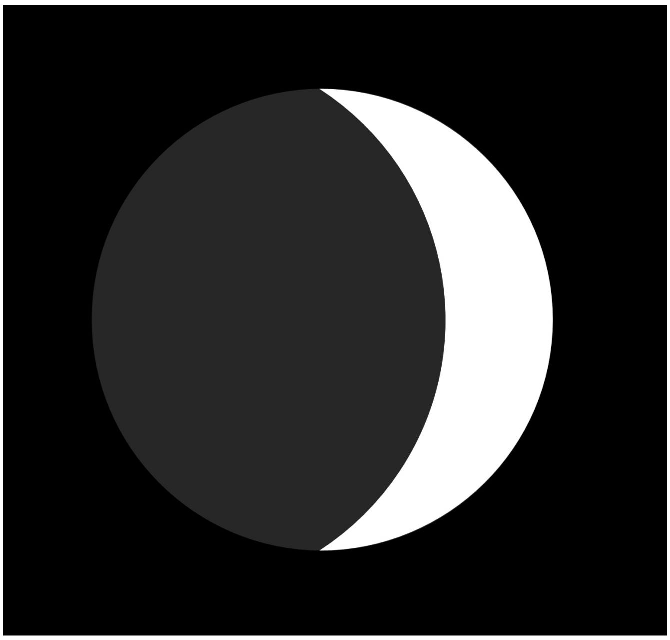

Figure 10. A crescent of the moon.

After college studies at Brussels with in particular Adolphe Quetelet as professor of mathematics, Plateau obtained his doctoral degree at Liege in 1829 with a thesis on human vision. Afterwards he was appointed as professor at Gent where he set up a great laboratory which a.o. was specialised for the scientific study of vision. The *Museum for the History of Science* at Gent could be highly recommended to be visited in this and many other respects, as, according to Georges Sarton, *the history of mathematics is the kernel of the history of human culture, the skeleton which supports and keeps together all the rest of the sciences;* ([18] offers a recent brief history of geometry).

About irradiation, in his *"Handbuch der Physiologischen Optik"*, (cfr. e.g. the third edition, 1911), Herman Helmholtz wrote as follows: *"Die Erscheinungen, welche Plateau als Irradiation beschreibt, ..., während sich alles einfach erklärt, wenn man annimmt, die Irradiation rühre Zerstreuungsbildern her."*; (in this handbook, also, a.o. the distinct opinions of Plateau, Fechner and Helmholtz on irradiation are put in a historical context). And, upon the latter statement, Plateau reacted ("a bit sharply" in the opinion of Dorikens) with *"Tout s'explique, mais de quel manière?"*.

In the present article, strolling somewhat further along the paths of our previous papers (cfr., in particular [1], [2], [3]), *to answer this question is actually precisely what we try to do:* to properly describe such a manner, at least qualitatively. Incidentally, close to the previous by the way, the first director of the *"Laboratory of Experimental Psychology and Paedagogy"* at Leuven was Armand Thiéry. After a.o. obtaining a doctor's degree in mathematics and physics at Leuven, he went to Leipzig, where in the mean time also Wundt had started a laboratory for experimental psychology, and made one more doctoral thesis there which offered a critical survey on the knowledge till about 1895 on visual illusions. For instance, Thiéry casually remarked that, at the time of his thesis, concerning Zöllner's illusion of 1860 there were already more than a thousand "explanations" going around ... . And, what were then and are up till now called "explanations" of the optical visual illusions, are, in many cases, at least in our opinion, rather evocations of certain situations which really may lead to analogous effects as the ones seen in these illusions proper, or, in many cases, are simply fantastic nonsense. And, in any case, still at least in our opinion, they have no connection whatsoever with any even remotely scientifical approach to the problem; as the natural scientist Marcel Minnaert of Gent and of Utrecht did formulate: *"Fantasie en rede, samenwerkend, elkaar aanvullend en doordringend,* *dat is ware wetenschap!"*. From Helmholtz's *"The Aim and Progress of Physical Science"* we quote the following: "Isolated facts and experiments have in themselves no value, however great their number may be. They only become valuable in a theoretical or practical point of view when they make us acquainted with the *law* of a series of uniformly recurring phenomena, or, it may be, only give a negative result showing an incompleteness in our own knowledge of such a law, till then held on to be perfect. ... *To find the law by which they are regulated is to understand phenomena"*. And as René Thom said in *"Paraboles et Catastrophes"*: *"Comprendre signifie avant tout géométriser!"*. Finally, at this stage, here is the opening sentence of Gustav Fechner's *"Elemente der Psychophysik"*: "Unter *Psychophysik* verstehe ich gemäss der, ... Erklärung eine Lehre, die, obwohl der Aufgabe nach uralt, doch in Betreff der Fassung und Behandlung dieser Aufgabe sich hier insoweit als eine neue darstellt, das man den neuen Namen dafür nicht unpassend und nicht unnöthig finden dürfte, kurz *eine exacte Lehre von den Beziehungen zwischen Leib und Seele"*. And, to make sure: our views as presented above in Section 1 at most claim to be a beginning of a *scientific understanding* of the concepts of human (visual) sensation and perception, i.e. to be the outline of a geometrical phenomenology for these processes.

As expressed by Chern in his foreword to the *"Handbook of Differential Geometry, Volume 1"* (Eds. Franki Dillen e.a., A'dam, 2000): *"While algebra and analysis provide the foundations of mathematics, geometry is at the core."*, and as stressed by Marcel Berger: *"the numero uno geometrical invariant is curvature"* (cfr. a.o. his *"A Panoramic view of Riemannian Geometry"* of 2003 and his *"Géométrie vivante"* of 2009). Two of the most well known examples of scientific understandings, or still, of geometric definitions, of some other fundamental natural concepts were discussed as such by Erwin Schrödinger in the article *"The General Theory of Relativity and Wave Mechanics"* (which he wrote at Gent for the 1940 volume of the Flemish "Wis– en Natuurkundig Tijdschrift"): *"The most important discoveries are those which in the course of time tend to become tautological.* The logical content of Newton's first two laws of motion was to state, that a body moves uniformly in a straight line, unless it does something else and that in the latter case we agree upon *calling force* its acceleration (i.e. basically, the *curvature vector field* of its trajectory in space, –the authors–) multiplied by an individual constant. The great achievement was, to concentrate attention on the *second* derivatives–to suggest that they–not the first or third or fourth, not any other property of the motion–ought to be accounted for by the environment. The fundamental statements of Einstein's theory of gravitation are of the similar kind. The equations $S_{hk} - \frac{1}{2}\tau g_{hk} = -8\pi T_{hk}$ (*) state, that the contracted curvature tensor S_{hk}, (or, the *Ricci tensor*, while τ and g_{hk} denote the *Riemannian scalar curvature* and the metric, respectively–the authors–), is either zero or not and that, when and where it's not, we *call matter* (T_{hk}) the left hand side of equations (*).". And, in a way, similarly, *we above called perception the (Casorati) curvature of sensation.*

5. Appendices

5.1. ON SOME RELATED ASPECTS OF GEOMETRY

In the consideration of relief surfaces in $R^3 = R^2 \times R$, whereby on the z–axis are extended, say, the values $z = F(x, y) \in R$ of sensations of lumonosities at the locations $(x, y) \in R^2$ in the image plane, *the co–ordinates* (x, y) *on the one hand and the* z *co–ordinates on the other hand do relate to by nature completely different quantities per sé and even are incommensurable.* Therefore, the use, in particular, of the *Euclidean geometry of the ambient space* R^3 to study the geometrical properties of these graph surfaces $z = F(x, y)$ *in principle is not acceptable* at all. Rather, in the present situation, the more appropriate geometry to be used on the standard space $R^3(x, y, z)$ would be the *1–fold isotropical geometry* which is determined by the *degenerate Riemannian first or metrical fundamental form* $g = ds^2 = dx^2 + dy^2$; (a fundamental article in this respect was written by Jan Koenderink and Andrea van Doorn, at Utrecht, not so long ago, [11]). For graphs $z = F(x, y)$ *in the ambient space* R^3 *structured by this latter geometry*, the *second fundamental form* h essentially is nothing but the (symmetrical) Hessian matrix of F, i.e.

$$H = \begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{pmatrix},$$

whereby the indices x and y hereby refer to partial differentiation. And, thus, the *Casorati curvatures* of such surfaces $z = F(x, y)$ in $(R^3(x, y, z), ds^2 = dx^2 + dy^2)$ are given by $\mathcal{C} = F_{xx}^2 + F_{xy}^2 + F_{yx}^2 + F_{yy}^2$; (cfr., most in particular at this stage, the above quote of Schrödinger).

In Figure 11 are shown some step functions $I(x)$ and their smoothings $I(x, s)$, their logarithms $\ln I(x, s)$ and their associated Fechner functions $F(x) = k \ln I(x, s)$ and in each case the curvatures $\mathcal{C}(x) = [f''(x)]^2$ of these functions $f : R \to R : x \mapsto f(x)$, in order to get some 1D–impression (of the 2D–situation which actually describes sensation and perception) of the geometry which is determined by such a degenerate metric. And, we draw the reader's attention to the following fact: the basic effects hereby observed in the shifts of the curvature extrema turn out to be very similar indeed to the ones observed before in the Euclidean framework which was shown in Figure 1.

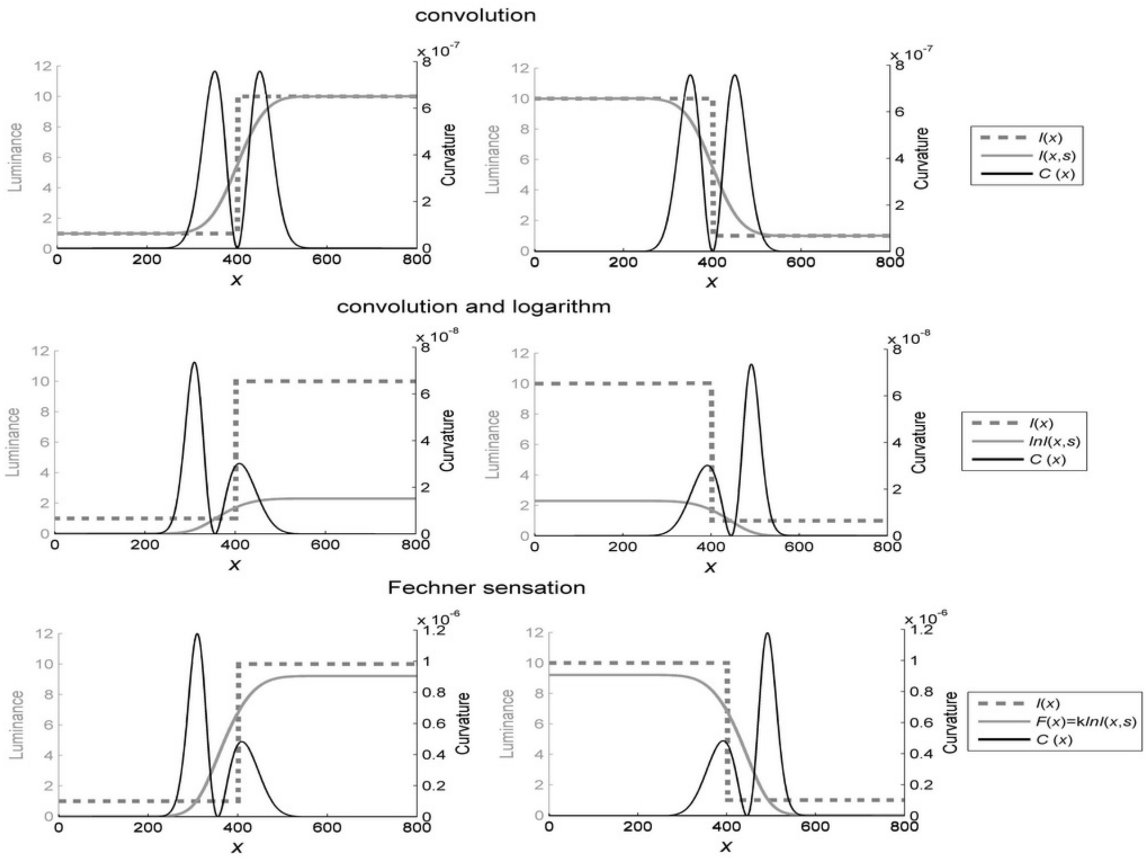

FIGURE 11. The curvature $\mathcal{C}(x) = [f''(x)]^2$ of planar graphs of functions $f : R \to R$.

And, in Figure 12, pictures are shown corresponding to dealing with the 1–*fold isotropic geometry* on $R^3 = R^2 \times R$ in the determination of the *curvatures of the Fechner surfaces* $z = F(x, y)$ yielded by the *images of the Helmholtz illusion.*

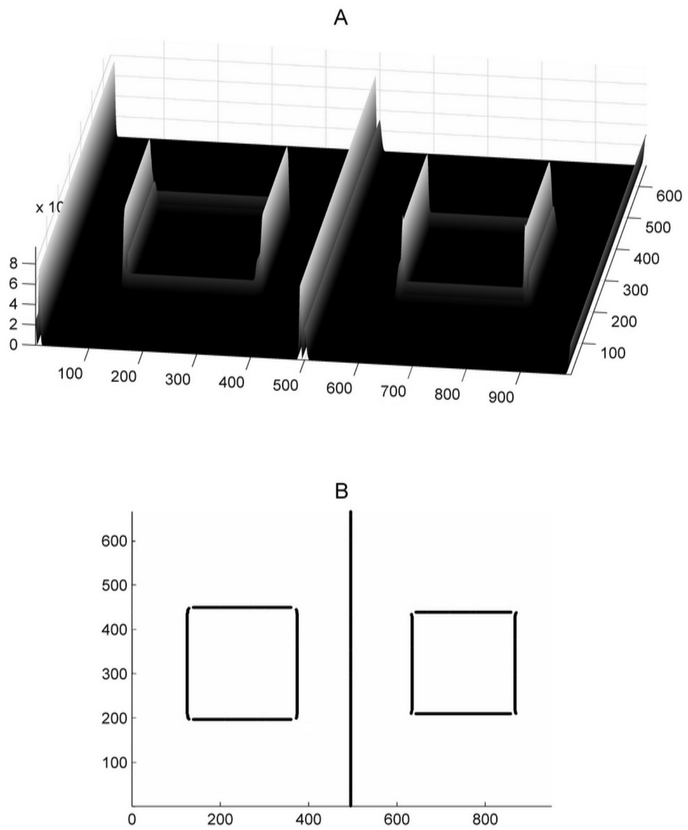

FIGURE 12. The Helmholtz irradiation illusion: the Casorati curvature surfaces $z = F_{xx}^2 + F_{xy}^2 + F_{yx}^2 + F_{yy}^2$ of the Fechner surfaces $z = F(x, y)$ corresponding to the given white and black squares, respectively, and the rectangles determined by their extrema.

Finally, in Figure 13, we show the analogous pictures for Kitaoka's bulge illusion. So, at least qualitatively, and for the present kind of considerations, the Euclidean geometrical structure $ds^2 = dx^2 + dy^2 + dz^2$ (cfr. the theorem of Pythagoras) and the 1–fold isotropical structure $ds^2 = dx^2 + dy^2$ (which geometrically cannot help to occur in the description of human visual sensation) on the standard space $R^3(x, y, z)$ basically measure the same effect pretty much in the same way. Therefore, almost just like that, before we have been so free to stage our geometrical descriptions of visual sensation and perception in the for all after all most familiar geometrical setting, i.e.: right from the start we have taken liberty to formulate our description of human vision within the framework of Euclidean geometry. (But also for all, it could be very worthwhile to read the classical textbook at present on "isotropical space geometry", namely Hans Sachs' "Isotrope Geometrie des Raumes" [19].)

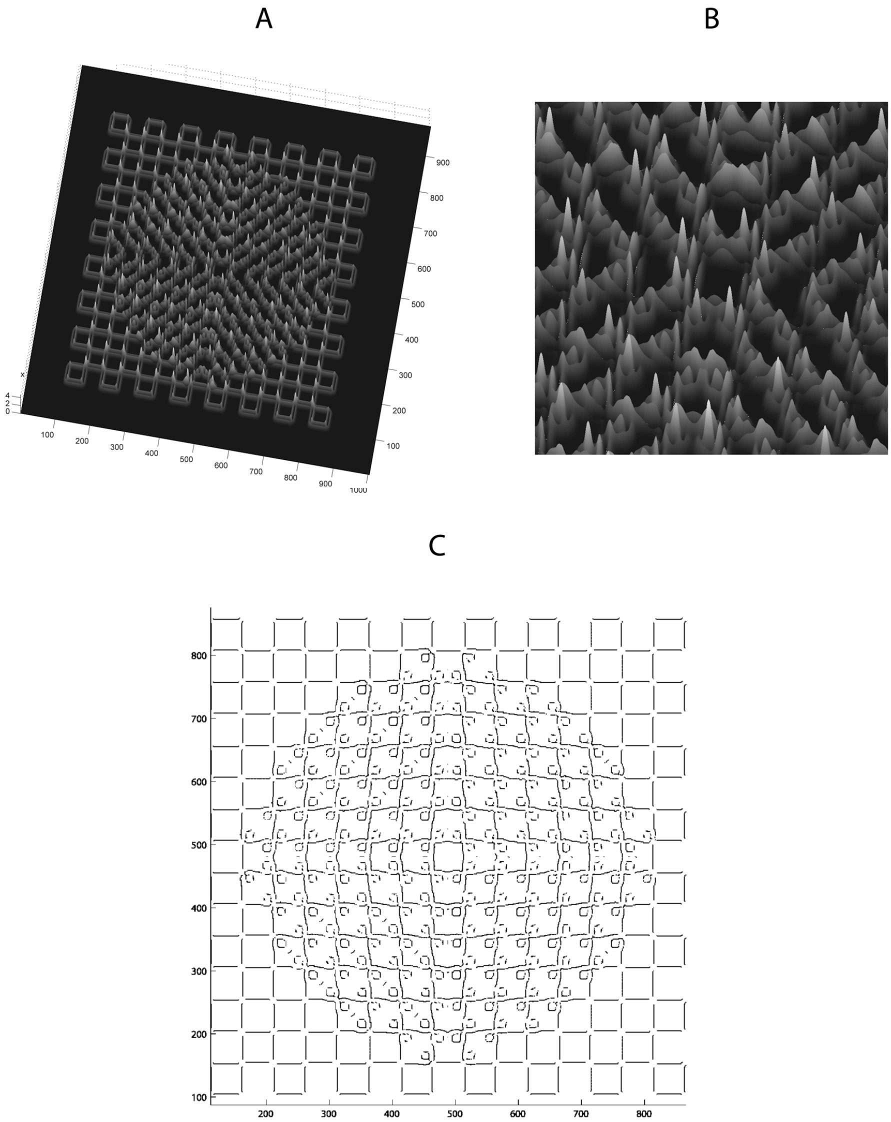

FIGURE 13. The Kitaoka bulge illusion: (A), a view of the Casorati curvature surfaces $z = F_{xx}^2 + F_{xy}^2 + F_{yx}^2 + F_{yy}^2$ of the Fechner surfaces $z = F(x, y)$; (B), a central close–up of this curvature surface; and (C), this Casorati surface's main extrema.

5.2. ON SOME RELATED ASPECTS OF FECHNER'S LAW

Many try to sketch a person's internal organisation by measuring brain activity, while only few attempt to understand the interaction between the external environment and the mind, that is, the conscious experience induced by external stimuli.

Philosophically, the body–mind problem has been stated in different ways and observed from different perspectives. Gustav Fechner revised the body–mind problem in a scientific approach, in which he mainly resisted materialism as a dominant stream of thought where physical events or physiology were treated in isolation from the mysterious mind. The experience accompanying an observation is what Fechner denoted as a sensation. In "Elemente der Psychophysik" [8], Fechner assumed that sensation could be measured hypothetically along an internal scale. To arrive at the elementary transformation between external stimulus magnitudes and sensation, he founded his law on two premises: (i) the *empirical law of Weber* $\Delta I/I = c$, for some constant c: when ΔI denotes the smallest difference that a person can notice between two physical stimulus quantities (e.g. like light intensity), then Weber's law states that the just–noticeable–difference is not constant along its physical scale, but it is proportional to the absolute level on this scale; and (ii) the *mathematical auxiliary principle*: infinitesimally small increments are proportional to observable increments. When we denote the steps on the hypothetical internal scale of sensations proposed by Fechner as ΔF, then ΔF is assumed to be constant ($= b$) along its range if the administered pairs of stimulus levels are committed to the constant ratio $\Delta I/I$, and by joining together both expressions, we come to the following relation: $\Delta F/(\Delta I/I) = b/c = k$ (**). By exerting Fechner's mathematical auxiliary principle, (**) becomes $dF = k(dI/I)$, and integrating yields $F = k \ln I + d$, whereby d is an integration constant (which can be omitted because the origin of the internal scale can be chosen appropriately).

To accommodate with some deviating empirical measurements, especially in the lower part of the scale, Weber's law can be generalized to $\Delta I = cI + a$, whereby c and a are constants, leading to $F = K \ln(cI + a)$ for some constant K, as a generalised Fechner's law. And further different versions have been put forward choosing some alternatives for the previous assumptions (e.g., see [20]). The internal scale cannot be accessed directly through measurements and psychophysical measurements are constrained to relate the induced stimulus magnitudes expressed in some physical quantity to a person's subjective report of stimulus magnitudes on some scale. For instance, assuming that $\Delta F/F$ is constant, just like $\Delta I/I$ was assumed to be constant in Weber's law, leads to two logarithmic scales for the physical and the sensory quantities concerned. Putting $\Delta F/F = n\Delta I/I$, whereby n denotes one more constant, by integration, the physical magnitude I and the corresponding sensation F can be related by *the power law* as $\ln F = \ln I^n + \ln L$, for some integration constant L. In the case of human vision, in practice, the power law $F = LI^n$ holds for some constant $n < 1$ and consequently the curvature behaviours of the sensation surfaces which correspond to given luminosities essentially are the same, whether using Fechner's law or using the power law. The debate between both laws is an empirical one and as further references in this context, in general and also to the point, see e.g. [21] and [22], [23]. In the present study, we defined human visual sensation in two steps by using the logarithm and convolution of light intensities. In the future, we intend to treat the mathematical description of human visual sensation in a more delicate manner.

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[1] Laboratory of Experimental Psychology,
Katholieke Universiteit Leuven,
Belgium
E-mail address: Bart.Ons@psy.kuleuven.be

[2] Section of Geometry,
Katholieke Universiteit Leuven,
Belgium
E-mail address: Leopold.Verstraelen@wis.kuleuven.be