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CHARACTERIZING ORDERED QUASI-IDEALS OF ORDERED Γ-SEMIGROUPS

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Abstract. The motivation mainly comes from the conditions of (ordered) quasi-ideals to be (0-)minimal or maximal that are of importance and interest in (ordered) semigroups. In 1981, the concept and notion of a Γ-semigroup was introduced by Sen [10]. We can see that any semigroup can be reduced to a Γ-semigroup. In this article, we give some auxiliary results are also necessary for our considerations and characterize the relationship between (0-)minimal and maximal ordered quasi-ideals in ordered Γ-semigroups and Q-simple and 0-Q-simple ordered Γ-semigroups analogous to the characterizations of (0-)minimal and maximal ordered quasi-ideals in ordered semigroups considered by Iampan [5].

1. Introduction

Let S be a semigroup. A subsemigroup Q of S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$. The definitions of a minimal quasi-ideal and a 0-minimal quasi-ideal in semigroups in [17] are given differently as follows: A quasi-ideal Q of a semigroup S without zero is called a *minimal quasi-ideal* of S if Q does not properly contain any quasi-ideal of S. For a semigroup S with zero, a *0-minimal quasi-ideal* of S is a nonzero quasi-ideal of S which does not properly contain any nonzero quasi-ideal of S. The notion of a quasi-ideal in semigroups was first introduced by Steinfeld [15] in 1956, and it has been widely studied. In 1956, Steinfeld [16] gave some characterizations of 0-minimal quasi-ideals in semigroups. In 1981, the concept and notion of

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an ordered Γ-semigroup was introduced by Sen [10]. In 1998, the concept and notion of an ordered quasi-ideal in ordered semigroups was introduced by Kehayopulu [7] as follows: Let S be an ordered semigroup. A subsemigroup Q of S is called an *ordered quasi-ideal* of S if $(SQ] \cap (QS] \subseteq Q$, and $x \in Q, (x] \subseteq Q$. In 2000, Cao and Xu [2] characterized the minimal and maximal left ideals in ordered semigroups, and gave some characterizations of minimal and maximal left ideals in ordered semigroups. In 2002, Arslanov and Kehayopulu [1] gave some characterizations of minimal and maximal ideals in ordered semigroups. In 2004, Iampan and Siripitukdet [6] characterized the (0-)minimal and maximal ordered left ideals in ordered Γ-semigroups, and gave some characterizations of (0-)minimal and maximal ordered left ideals in ordered Γ-semigroups. In 2006, the concept and notion of a quasi-ideal in Γ-semigroups was introduced by Chinram [3]. In 2008, Iampan [5] characterized the (0-)minimal and maximal ordered quasi-ideals in ordered semigroups and gave some characterizations of (0-)minimal and maximal ordered quasi-ideals in ordered semigroups.

The concept of a (0-)minimal and maximal one-sided ideal or ideal is the really interested and important thing in (ordered) semigroups and (ordered) Γ-semigroups. We can see that the notion of a one-side ideal is a generalization of the notion of an ideal, and the notion of a quasi-ideal is a generalization of the notion of a one-side ideal. Hence we also characterize the (0-)minimal and maximal ordered quasi-ideals in ordered Γ-semigroups, and give some characterizations of (0-)minimal and maximal ordered quasi-ideals in ordered Γ-semigroups.

To present the main theorems we first recall the definition of a Γ-semigroup which is important here.

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [10] if there exists a mappings $M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. We can see that any semigroup can be considered as a Γ-semigroup. A nonempty subset K of a Γ-semigroup M is called a *sub*-Γ-*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. For nonempty subsets A, B of M, let $A\Gamma B := \{a\gamma b : a \in A, b \in B \text{ and } \gamma \in \Gamma\}$. We also write $a\Gamma B, A\Gamma b$ and $a\Gamma b$ for $\{a\}\Gamma B, A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively.

Examples of Γ-semigroups can be seen in [6, 9, 11] and [12] respectively.

The following example comes from Dixit and Dewan [4].

Example 1.1. Let $T = \{-i, 0, i\}$ and $\Gamma = T$. Then T is a Γ-semigroup under the multiplication over complex number while T is not a semigroup under complex number multiplication.

A partially ordered Γ-semigroup M is called an *ordered* Γ*-semigroup* (some authors called po-Γ-semigroup) if for any $a, b, c \in M$ and $\gamma \in \Gamma$,

$$a \leq b \text{ implies } a\gamma c \leq b\gamma c \text{ and } c\gamma a \leq c\gamma b.$$

If $(M; \leq)$ is an ordered Γ-semigroup, and K is a sub-Γ-semigroup of M, then $(K; \leq)$ is an ordered Γ-semigroup. For an element a of an ordered Γ-semigroup M, define $(a] = \{t \in M : t \leq a\}$ and for a subset H of M, define $(H] = \bigcup\limits_{h \in H} (h]$, that is, $(H] = \{t \in M : t \leq h \text{ for some } h \in H\}$, and $H \cup a := H \cup \{a\}$. We observe here that

1. $H \subseteq (H] = ((H]]$.
2. For any subsets A and B of M with $A \subseteq B$, we have $(A] \subseteq (B]$.
3. For any subsets A and B of M, we have $(A \cup B] = (A] \cup (B]$.
4. For any subsets A and B of M, we have $(A \cap B] \subseteq (A] \cap (B]$.
5. For any elements a and b of M with $a \leq b$, we have $(a\Gamma M] \subseteq (b\Gamma M]$ and $(M\Gamma a] \subseteq (M\Gamma b]$.

Examples of ordered Γ-semigroups can be seen in [6] and [14] respectively.

The following definitions are introduced analogously to some definitions in [7].

A nonempty subset I of an ordered Γ-semigroup M is called an *ordered ideal* of M if $M\Gamma I \subseteq I$ and $I\Gamma M \subseteq I$, and for any $x \in I, (x] \subseteq I$. Then for any $a \in M$, we have $(M\Gamma a]$ is an ordered left ideal of M, and $(a\Gamma M]$ is an ordered right ideal of M [6]. A sub-Γ-semigroup Q of an ordered Γ-semigroup M is called an *ordered quasi-ideal* of M if $(M\Gamma Q] \cap (Q\Gamma M] \subseteq Q$, and for any $x \in Q, (x] \subseteq Q$. Then the notion of an ordered quasi-ideal is a generalization of the notion of an ordered ideal. The intersection of all ordered quasi-ideals of a sub-Γ-semigroup K of an ordered Γ-semigroup M containing a nonempty subset A of K is called the *ordered quasi-ideal of K generated by A*. For $A = \{a\}$, let $Q_K(a)$ denote the ordered quasi-ideal of K generated by $\{a\}$. If $K = M$, then we also write $Q_M(a)$ as $Q(a)$. An element a of an ordered Γ-semigroup M with at least two elements is called a *zero element* of M if $x\gamma a = a\gamma x = a$ for all $x \in M$ and $\gamma \in \Gamma$, and $a \leq x$ for all $x \in M$, and denote it by 0. If M is an ordered Γ-semigroup with zero, then every ordered quasi-ideal of M containing a zero element. An ordered Γ-semigroup M without zero is called *Q-simple* if it has no proper ordered quasi-ideals. An ordered Γ-semigroup M with zero is called *0-Q-simple* if it has no nonzero proper ordered quasi-ideals and $M\Gamma M \neq \{0\}$. An ordered quasi-ideal Q of an ordered Γ-semigroup M without zero is called a *minimal ordered quasi-ideal* of M if there is no ordered quasi-ideal A of M such that $A \subset Q$. Equivalently, if for any ordered quasi-ideal A of M such that $A \subseteq Q$, we have $A = Q$. A nonzero ordered quasi-ideal Q of an ordered Γ-semigroup M with zero is called a *0-minimal ordered quasi-ideal* of M if there is no nonzero ordered quasi-ideal A of M such that $A \subset Q$. Equivalently, if for any nonzero ordered quasi-ideal A of M such that $A \subseteq Q$, we have $A = Q$. Equivalently, if for any ordered quasi-ideal A of M such that $A \subset Q$, we have $A = \{0\}$. A proper ordered quasi-ideal Q of an ordered Γ-semigroup M is called a *maximal ordered quasi-ideal* of M if for any ordered quasi-ideal A of M such that $Q \subset A$, we have $A = M$. Equivalently, if for any proper ordered quasi-ideal A of M such that $Q \subseteq A$, we have $A = Q$.

Our aim in this paper is fourfold.

1. To introduce the concept of a Q-simple ordered Γ-semigroup and a 0-Q-simple ordered Γ-semigroup.
2. To characterize the properties of ordered quasi-ideals in ordered Γ-semigroups.
3. To characterize the relationship between (0-)minimal ordered quasi-ideals and (0-)Q-simple ordered Γ-semigroups.
4. To characterize the relationship between maximal ordered quasi-ideals and (0-)Q-simple ordered Γ-semigroups.

2. Lemmas

We shall assume throughout this paper that M stands for an ordered Γ-semigroup. Before the characterizations of ordered quasi-ideals for the main results, we give some auxiliary results which are necessary in what follows. The following three lemmas are also necessary for our considerations and easy to verify.

Lemma 2.1. *For any nonempty subset A of M, $(((M\Gamma A] \cap (A\Gamma M]) \cup A]$ is the smallest ordered quasi-ideal of M containing A.*
Furthermore, for any $a \in M$,

$$Q(a) = (((M\Gamma a] \cap (a\Gamma M]) \cup a] = ((M\Gamma a] \cap (a\Gamma M]] \cup (a].$$

Lemma 2.2. *The set $((M\Gamma a] \cap (a\Gamma M]]$ is an ordered quasi-ideal of M for all $a \in M$.*

Lemma 2.3. *Let $\{Q_\gamma : \gamma \in \Lambda\}$ be a collection of ordered quasi-ideals of M. Then $\bigcap_{\gamma\in\Lambda} Q_\gamma$ is an ordered quasi-ideal of M if $\bigcap_{\gamma\in\Lambda} Q_\gamma \neq \emptyset$.*

Lemma 2.4. *If M has no zero element, then the following statements are equivalent:*

(i) *M is Q-simple.*

(ii) *$((M\Gamma a] \cap (a\Gamma M]] = M$ for all $a \in M$.*

(iii) *$Q(a) = M$ for all $a \in M$.*

Proof. Since M is Q-simple, we have $((M\Gamma a] \cap (a\Gamma M]] = M$ for all $a \in M$ by Lemma 2.2. Therefore (i) implies (ii). By Lemma 2.1, we have $Q(a) = ((M\Gamma a] \cap (a\Gamma M]] \cup (a] = M \cup (a] = M$ for all $a \in M$. Thus (ii) implies (iii). Now, let Q be an ordered quasi-ideal of M, and let $a \in Q$. Then $M = Q(a) \subseteq Q \subseteq M$, so $Q = M$. Hence M is Q-simple, we have that (iii) implies (i). □

Lemma 2.5. *If M has a zero element, then the following statements hold:*

(i) *If M is 0-Q-simple, then $Q(a) = M$ for all $a \in M \setminus \{0\}$.*

(ii) *If $Q(a) = M$ for all $a \in M\setminus\{0\}$, then either $M\Gamma M = \{0\}$ or M is 0-Q-simple.*

Proof. (i) Assume that M is 0-Q-simple. Then, since for any $a \in M \setminus \{0\}$, $Q(a)$ is a nonzero ordered quasi-ideal of M, we have $Q(a) = M$ for all $a \in M \setminus \{0\}$.

(ii) Assume that $Q(a) = M$ for all $a \in M \setminus \{0\}$ and $M\Gamma M \neq \{0\}$. Now, let Q be a nonzero ordered quasi-ideal of M, and let $a \in Q \setminus \{0\}$. Then $M = Q(a) \subseteq Q \subseteq M$, so $Q = M$. Therefore M is 0-Q-simple. □

Lemma 2.6. *If Q is an ordered quasi-ideal of M, and K is a sub-Γ-semigroup of M, then the following statements hold:*

(i) *If K is Q-simple such that $K \cap Q \neq \emptyset$, then $K \subseteq Q$.*

(ii) *If K is 0-Q-simple such that $K \setminus \{0\} \cap Q \neq \emptyset$, then $K \subseteq Q$.*

Proof. (i) Assume that K is Q-simple with $K \cap Q \neq \emptyset$, and let $a \in K \cap Q$. Then, by Lemma 2.2, $((K\Gamma a] \cap (a\Gamma K]] \cap K$ is an ordered quasi-ideal of K. Hence $((K\Gamma a] \cap (a\Gamma K]] \cap K = K$. Therefore $K \subseteq ((K\Gamma a] \cap (a\Gamma K]] \subseteq ((M\Gamma Q] \cap (Q\Gamma M]] \subseteq (Q] = Q$, we conclude that $K \subseteq Q$.

(ii) Assume that K is 0-Q-simple such that $K\setminus\{0\}\cap Q \neq \emptyset$, and let $a \in K\setminus\{0\}\cap Q$. Then, by Lemmas 2.1 and 2.5 (i), we have $K = Q_K(a) = (((K\Gamma a] \cap (a\Gamma K]] \cup a] \cap K \subseteq (((K\Gamma a] \cap (a\Gamma K]] \cup a] \subseteq (((M\Gamma a] \cap (a\Gamma M]] \cup a] = Q(a) \subseteq Q$. Therefore $K \subseteq Q$.

Hence the proof is completed. □

3. Main Results

The aim of this section is to characterize the relationship between minimal ordered quasi-ideals and Q-simple ordered Γ-semigroups, 0-minimal ordered quasi-ideals and 0-Q-simple ordered Γ-semigroups, and maximal ordered quasi-ideals and the set $\mathcal{U}$.

Theorem 3.1. *If M has no zero element, and Q is an ordered quasi-ideal of M, then the following statements hold:*

(i) *If Q is an ordered ideal of M and a minimal ordered quasi-ideal without zero of M, then either there exists an ordered quasi-ideal A of Q such that $Q\Gamma A \cap A\Gamma Q = \emptyset$ or Q is Q-simple.*

(ii) *If Q is Q-simple, then Q is a minimal ordered quasi-ideal of M.*

(iii) *If Q is an ordered ideal of M and a minimal ordered quasi-ideal with zero of M, then either there exists a nonzero ordered quasi-ideal A of Q such that $Q\Gamma A \cap A\Gamma Q = \{0\}$ or Q is 0-Q-simple.*

Proof. (i) Assume that an ordered ideal Q is a minimal ordered quasi-ideal without zero of M, and let $Q\Gamma A \cap A\Gamma Q \neq \emptyset$ for all ordered quasi-ideals A of Q. Clearly, Q is a sub-Γ-semigroup of M. Now, let A be an ordered quasi-ideal of Q. Then $\emptyset \neq Q\Gamma A \cap A\Gamma Q \subseteq (Q\Gamma A] \cap (A\Gamma Q] \subseteq A$. Define $H := \{h \in A : h \in (Q\Gamma A] \cap (A\Gamma Q]\}$. Then $\emptyset \neq H \subseteq A \subseteq Q$. To show that H is an ordered quasi-ideal of M, let $h_1, h_2 \in H$ and $\gamma \in \Gamma$. Then $h_1 \leq q_1\beta_1 a_1$ and $h_1 \leq a_1'\beta_1' q_1'$, and $h_2 \leq q_2\beta_2 a_2$ and $h_2 \leq a_2'\beta_2' q_2'$ for some $a_1, a_1', a_2, a_2' \in A, q_1, q_1', q_2, q_2' \in Q$ and $\beta_1, \beta_1', \beta_2, \beta_2' \in \Gamma$, so $h_1\gamma h_2 \leq q_1\beta_1 a_1\gamma q_2\beta_2 a_2$ and $h_1\gamma h_2 \leq a_1'\beta_1' q_1'\gamma a_2'\beta_2' q_2'$. Since A is an ordered quasi-ideal of Q, we get $A\Gamma Q\Gamma A \subseteq A$. Thus $a_1\gamma q_2\beta_2 a_2, a_1'\beta_1' q_1'\gamma a_2' \in A$. Since $h_1\gamma h_2 \in H\Gamma H \subseteq A\Gamma A \subseteq A$, we have $h_1\gamma h_2 \in H$. Hence H is a sub-Γ-semigroup of M. If $x \in (M\Gamma H] \cap (H\Gamma M]$, then $x \leq m\gamma h$ and $x \leq h'\gamma' m'$ for some $m, m' \in M, h, h' \in H$ and $\gamma, \gamma' \in \Gamma$. Thus $h \leq q_1\beta_1 a_1$ and $h \leq a_2\beta_2 q_2$, and $h' \leq q_1'\beta_1' a_1'$ and $h' \leq a_2'\beta_2' q_2'$ for some $a_1, a_1', a_2, a_2' \in A, q_1, q_1', q_2, q_2' \in Q$ and $\beta_1, \beta_1', \beta_2, \beta_2' \in \Gamma$. Hence $x \leq m\gamma h \leq m\gamma q_1\beta_1 a_1$ and $x \leq h'\gamma' m' \leq a_2'\beta_2' q_2'\gamma' m'$. Since Q is an ordered ideal of M, we have $m\gamma q_1, q_2'\gamma' m' \in Q$. Thus $x \in (Q\Gamma A] \cap (A\Gamma Q] \subseteq A$. Hence $x \in H$, so $(M\Gamma H] \cap (H\Gamma M] \subseteq H$. Next, let $x \in M$ and $h \in H$ be such that $x \leq h$. Then $x \in ((Q\Gamma A] \cap (A\Gamma Q]] \subseteq ((Q\Gamma A]] \cap ((A\Gamma Q]] = (Q\Gamma A] \cap (A\Gamma Q] \subseteq A$. Hence $x \in H$, so H is an ordered quasi-ideal of M. Since Q is a minimal ordered quasi-ideal of M, we get $H = Q$. Therefore $A = Q$, we conclude that Q is Q-simple.

(ii) Assume that Q is Q-simple, and let A be an ordered quasi-ideal of M such that $A \subseteq Q$. Then $A \cap Q \neq \emptyset$, it follows from Lemma 2.6 (i) that $Q \subseteq A$. Hence $A = Q$, so Q is a minimal ordered quasi-ideal of M.

(iii) It is similar to the proof of statement (i).

Therefore we complete the proof of the theorem. □

Using the same proof of Theorem 3.1 (i) and Lemma 2.6 (ii), we have Theorem 3.2.

Theorem 3.2. *If M has a zero element, and Q is a nonzero ordered quasi-ideal of M, then the following statements hold:*

(i) *If Q is an ordered ideal of M and a 0-minimal ordered quasi-ideal of M, then either there exists a nonzero ordered quasi-ideal A of Q such that $Q\Gamma A \cap A\Gamma Q = \{0\}$ or Q is 0-Q-simple.*

(ii) *If Q is 0-Q-simple, then Q is a 0-minimal ordered quasi-ideal of M.*

Theorem 3.3. *If M has no zero element but it has a proper ordered quasi-ideal, then every proper ordered quasi-ideal of M is minimal if and only if the intersection of any two distinct proper ordered quasi-ideals is empty.*

Proof. Assume Q_1 and Q_2 are two distinct proper ordered quasi-ideals of M. Then Q_1 and Q_2 are minimal ordered quasi-ideals of M. If $Q_1 \cap Q_2 \neq \emptyset$, then $Q_1 \cap Q_2$ is an ordered quasi-ideal of M by Lemma 2.3. Since Q_1 and Q_2 are minimal ordered quasi-ideals of M, we get $Q_1 = Q_2$. It is impossible. Therefore $Q_1 \cap Q_2 = \emptyset$.

The converse is obvious. □

Using the same proof of Theorem 3.3, we have Theorem 3.4.

Theorem 3.4. *If M has a zero element and a nonzero proper ordered quasi-ideal, then every nonzero proper ordered quasi-ideal of M is 0-minimal if and only if the intersection of any two distinct nonzero proper ordered quasi-ideals is $\{0\}$.*

Theorem 3.5. *Assume Q is an ordered quasi-ideal of M. If either $M \setminus Q = \{a\}$ for some $a \in M$ or $M \setminus Q \subseteq ((M\Gamma b] \cap (b\Gamma M]]$ for all $b \in M \setminus Q$, then Q is a maximal ordered quasi-ideal of M.*

Proof. Let A be an ordered quasi-ideal of M such that $Q \subset A$. Then we consider the following two cases:

Case 1: $M \setminus Q = \{a\}$ for some $a \in M$.

Then $M = Q \cup \{a\}$. Since $Q \subset A$, we have $\emptyset \neq A \setminus Q \subseteq M \setminus Q = \{a\}$. Hence $A \setminus Q = \{a\}$, so $A = Q \cup \{a\} = M$.

Case 2: $M \setminus Q \subseteq ((M\Gamma b] \cap (b\Gamma M]]$ for all $b \in M \setminus Q$.

If $b \in A\setminus Q \subseteq M\setminus Q$, then $M\setminus Q \subseteq ((M\Gamma b]\cap(b\Gamma M]] \subseteq ((M\Gamma A]\cap(A\Gamma M]] \subseteq (A] = A$. Therefore $M = Q \cup M \setminus Q \subseteq Q \cup A = A$, so $A = M$.

Hence we conclude that Q is a maximal ordered quasi-ideal of M. $\square$

Theorem 3.6. *If Q is a maximal ordered quasi-ideal of M, and $Q \cup Q(a)$ is an ordered quasi-ideal of M for all $a \in M \setminus Q$, then either*

(i) *$M \setminus Q \subseteq (a]$ and $a\Gamma a \subseteq Q$ for some $a \in M \setminus Q$, and $((M\Gamma b] \cap (b\Gamma M]] \subseteq Q$ for all $b \in M \setminus Q$ or*

(ii) *$M \setminus Q \subseteq Q(a)$ for all $a \in M \setminus Q$.*

Proof. Assume that Q is a maximal ordered quasi-ideal of M, and $Q \cup Q(a)$ is an ordered quasi-ideal of M for all $a \in M \setminus Q$. Then we have the following two cases:

Case 1: $((M\Gamma a] \cap (a\Gamma M]] \subseteq Q$ for some $a \in M \setminus Q$.

Then $a\Gamma a \subseteq M\Gamma a \cap a\Gamma M \subseteq (M\Gamma a] \cap (a\Gamma M] \subseteq ((M\Gamma a] \cap (a\Gamma M]] \subseteq Q$, so $a\Gamma a \subseteq Q$. Since $Q \cup (a] = (Q \cup ((M\Gamma a] \cap (a\Gamma M]]) \cup (a] = Q \cup (((M\Gamma a] \cap (a\Gamma M]] \cup (a]) = Q \cup (((M\Gamma a] \cap (a\Gamma M]) \cup a] = Q \cup Q(a)$, we have $Q \cup (a]$ is an ordered quasi-ideal of M. Since $a \in M \setminus Q$, we get $Q \subset Q \cup (a]$. Thus $Q \cup (a] = M$ because Q is a maximal ordered quasi-ideal of M, so $M \setminus Q \subseteq (a]$. If $b \in M \setminus Q$, then $b \in (a]$. Thus $b \leq a$, so $((M\Gamma b] \cap (b\Gamma M]] \subseteq ((M\Gamma a] \cap (a\Gamma M]] \subseteq Q$. Hence $((M\Gamma b] \cap (b\Gamma M]] \subseteq Q$ for all $b \in M \setminus Q$. In this case, the condition (i) is satisfied.

Case 2: $((M\Gamma a] \cap (a\Gamma M]] \nsubseteq Q$ for all $a \in M \setminus Q$.

If $a \in M \setminus Q$, then $Q \subset Q \cup ((M\Gamma a] \cap (a\Gamma M]] \subseteq Q \cup Q(a)$ by Lemma 2.1. Since $Q \cup Q(a)$ is an ordered quasi-ideal of M, and Q is a maximal ordered quasi-ideal of M, we get $Q \cup Q(a) = M$. Therefore $M \setminus Q \subseteq Q(a)$ for all $a \in M \setminus Q$. In this case, the condition (ii) is satisfied.

Hence the proof is completed. $\square$

For an ordered Γ-semigroup M, let $\mathcal{U}$ denote the union of all nonzero proper ordered quasi-ideals of M if M has a zero element, and let $\mathcal{U}$ denote the union of all proper ordered quasi-ideals of M if M has no zero element. Then it is easy to verify Lemma 3.1.

Lemma 3.1. *$M = \mathcal{U}$ if and only if $Q(a) \neq M$ for all $a \in M$.*

As a consequence of Theorem 3.6 and Lemma 3.1, we obtain Theorem 3.7.

Theorem 3.7. *If M has no zero element, then one of the following four conditions is satisfied:*

(i) $\mathcal{U}$ *is not an ordered quasi-ideal of* M.

(ii) $Q(a) \neq M$ *for all* $a \in M$.

(iii) *There exists* $a \in M$ *such that* $Q(a) = M, (a] \nsubseteq ((M\Gamma a] \cap (a\Gamma M]]$ *and* $a\Gamma a \subseteq \mathcal{U}$, M *is not* Q*-simple,* $M \setminus \mathcal{U} = \{x \in M : Q(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered quasi-ideal of* M.

(iv) $M \setminus \mathcal{U} \subseteq Q(a)$ *for all* $a \in M \setminus \mathcal{U}$, M *is not* Q*-simple,* $M \setminus \mathcal{U} = \{x \in M : Q(x) = M\}$, *and* $\mathcal{U}$ *is the unique maximal ordered quasi-ideal of* M.

Proof. Assume that $\mathcal{U}$ is an ordered quasi-ideal of M. Then $\mathcal{U} \neq \emptyset$. Thus we consider the following two cases:

Case 1: $\mathcal{U} = M$.

By Lemma 3.1, we have $Q(a) \neq M$ for all $a \in M$. In this case, the condition (ii) is satisfied.

Case 2: $\mathcal{U} \neq M$.

Then M is not Q-simple. To show that $\mathcal{U}$ is the unique maximal ordered quasi-ideal of M, let A is an ordered quasi-ideal of M such that $\mathcal{U} \subset A$. If $A \neq M$, then A is a proper ordered quasi-ideal of M. Thus $A \subseteq \mathcal{U}$, so it is a contradiction. Hence $\mathcal{U}$ is a maximal ordered quasi-ideal of M. Next, assume that Q is a maximal ordered quasi-ideal of M. Then $Q \subseteq \mathcal{U} \subset M$ because Q is a proper ordered quasi-ideal of M. Since Q is a maximal ordered quasi-ideal of M, we have $Q = \mathcal{U}$. Hence $\mathcal{U}$ is the unique maximal ordered quasi-ideal of M. Since $\mathcal{U} \neq M$, it follows from Lemma 3.1 that $Q(x) = M$ for some $x \in M$. Obviously, $Q(x) = M$ for all $x \in M \setminus \mathcal{U}$. Thus $M \setminus \mathcal{U} = \{x \in M : Q(x) = M\}$, so $\mathcal{U} \cup Q(x) = M$ is an ordered quasi-ideal of M for all $x \in M \setminus \mathcal{U}$. By Theorem 3.6, we have the following two cases:

(i) $M \setminus \mathcal{U} \subseteq (a]$ and $a\Gamma a \subseteq \mathcal{U}$ for some $a \in M \setminus \mathcal{U}$, and $((M\Gamma b] \cap (b\Gamma M]] \subseteq \mathcal{U}$ for all $b \in M \setminus \mathcal{U}$ or

(ii) $M \setminus \mathcal{U} \subseteq Q(a)$ for all $a \in M \setminus \mathcal{U}$.

Assume that $M \setminus \mathcal{U} \subseteq (a]$ and $a\Gamma a \subseteq \mathcal{U}$ for some $a \in M \setminus \mathcal{U}$, and $((M\Gamma b] \cap (b\Gamma M]] \subseteq \mathcal{U}$ for all $b \in M \setminus \mathcal{U}$. If $(a] \subseteq ((M\Gamma a] \cap (a\Gamma M]]$, then $M = Q(a) = (((M\Gamma a] \cap (a\Gamma M]) \cup a] = ((M\Gamma a] \cap (a\Gamma M]]$. By hypothesis, $M = ((M\Gamma a] \cap (a\Gamma M]] \subseteq \mathcal{U}$ and so $\mathcal{U} = M$. It is impossible. Hence $(a] \nsubseteq ((M\Gamma a] \cap (a\Gamma M]]$. In this case, the condition (iii) is satisfied. Next, assume $M \setminus \mathfrak{U} \subseteq Q(a)$ for all $a \in M \setminus \mathfrak{U}$. In this case, the condition (iv) is satisfied.

Hence the proof is now completed. $\square$

Using the same proof of Theorem 3.7, we have Theorem 3.8.

Theorem 3.8. *If M has a zero element and $M\Gamma M \neq \{0\}$, then one of the following five conditions is satisfied:*

(i) *$\mathfrak{U}$ is not an ordered quasi-ideal of M.*
(ii) *$Q(a) \neq M$ for all $a \in M$.*
(iii) *$\mathfrak{U} = \{0\}, M \setminus \mathfrak{U} = \{x \in M : Q(x) = M\}$, and $\mathfrak{U}$ is the unique maximal ordered quasi-ideal of M.*
(iv) *There exists $a \in M$ such that $Q(a) = M, (a] \nsubseteq ((M\Gamma a] \cap (a\Gamma M]]$ and $a\Gamma a \subseteq \mathfrak{U}$, M is not 0-Q-simple, $M \setminus \mathfrak{U} = \{x \in M : Q(x) = M\}$, and $\mathfrak{U}$ is the unique maximal ordered quasi-ideal of M.*
(v) *$M \setminus \mathfrak{U} \subseteq Q(a)$ for all $a \in M \setminus \mathfrak{U}$, M is not 0-Q-simple, $M \setminus \mathfrak{U} = \{x \in M : Q(x) = M\}$, and $\mathfrak{U}$ is the unique maximal ordered quasi-ideal of M.*

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