Kragujevac Journal of Mathematics
Volume 35 Number 1 (2011), Pages 39–57.

THE χ^2 SEQUENCE SPACES DEFINED BY A MODULUS

NAGARAJAN SUBRAMANIAN [1], UMAKANTA MISRA [2], AND VLADIMIR RAKOČEVIĆ [3]

Abstract. In this paper we introduce the following sequence spaces
$\left\{x \in \chi^2 : P - \lim_{k,\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}}\right) = 0\right\}$
and $\left\{x \in \Lambda^2 : \sup_{k,\ell} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} a_{k\ell}^{mn} f\left(|x_{mn}|^{\frac{1}{m+n}}\right) < \infty\right\}$ where f is a modulus function and A is a nonnegative four dimensional matrix. We establish the inclusion theorems between these spaces and also general properties are discussed.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : \sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\},$$

Key words and phrases. Gai sequence, Analytic sequence, Modulus function, Double sequences.
2010 *Mathematics Subject Classification.* 40A05, 40C05, 40D05.
Received: August 19, 2010.
Revised: February 04, 2011.

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Başar [27] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Başar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p,q,u)$ of double sequences and have given some inclusion relations.

Spaces are strongly summable sequences was discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary (single) sequence spaces to multiply sequence spaces. This will be accomplished by presenting the following sequence spaces: $\left\{x \in \chi^2 : P - \lim_{k,\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(\left((m+n)! \left|x_{mn}\right|\right)^{\frac{1}{m+n}}\right) = 0\right\}$ and $\left\{x \in \Lambda^2 : \sup_{k,\ell} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} a_{k\ell}^{mn} f\left(\left|x_{mn}\right|^{\frac{1}{m+n}}\right) < \infty\right\}$ where f is a modulus function and A is a nonnegative four dimensional matrix. Other implications,general properties and variations will also be presented.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $\sup_{mn} \left|x_{mn}\right|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $\left((m+n)! \left|x_{mn}\right|\right)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{$all finite sequences$\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space (or a metric space) X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)\,(u \geq 0)$. The Δ_2- condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) X'= the continuous dual of X;
(ii) $X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\}$;
(iii) $X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent}, \text{ for each } x \in X \right\}$;
(iv) $X^{\gamma} = \left\{ a = (a_{mn}) : \sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{ for each } x \in X \right\}$;
(v) let X be an FK-space $\supset \phi$; then $X^{f} = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi) $X^{\delta} = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \right\}$.

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha-$ (or Köthe-Toeplitz) dual of $X, \beta-$ (or generalized-Köthe-Toeplitz) dual of $X, \gamma-$ dual of X, $\delta-$dual of X respectively. X^{α} is defined by Gupta and Kamptan [20]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by Başar and Altay in [42] and in the case $0 < p < 1$ by Altay and Başar in [43]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \ge 1} |\Delta x_k| \quad \text{and} \quad \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}, (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definitions and Preliminaries

χ^2_M and Λ^2_M denote the Pringscheims sense of double Orlicz space of gai sequences and Pringscheims sense of double Orlicz space of bounded sequences respectively.

Definition 2.1. A modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$;
(2) $f(x + y) \le f(x) + f(y)$, for all $x \ge 0$, $y \ge 0$;
(3) f is increasing;

(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that f is continuous on $[0, \infty)$.

Definition 2.2. Let p, q be semi norms on a vector space X. Then p is said to be stronger than q if whenever (x_{mn}) is a sequence such that $p(x_{mn}) \to 0$, then also $q(x_{mn}) \to 0$. If each is stronger than the others, the p and q are said to be equivalent.

Lemma 2.1. *Let p and q be semi norms on a linear space X. Then p is stronger than q if and only if there exists a constant M such that $q(x) \leq Mp(x)$ for all $x \in X$.*

Definition 2.3. A sequence E is said to be solid or normal if $(\alpha_{mn} x_{mn}) \in E$ whenever $(x_{mn}) \in E$ and for all sequences of scalars (α_{mn}) with $|\alpha_{mn}| \leq 1$, for all $m, n \in \mathbb{N}$.

Definition 2.4. A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 2.1. From the two above definitions it is clear that a sequence space E is solid implies that E is monotone.

Definition 2.5. A set E is said to be convergence free if $(y_{mn}) \in E$ whenever $(x_{mn}) \in E$ and $x_{mn} = 0$ implies that $y_{mn} = 0$.

By the gai of a double sequence we mean the gai on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Pringsheim limit 0 (denoted by $P - \lim x = 0$) such that $((m+n)! \, |x_{mn}|)^{\frac{1}{m+n}} \to 0$, whenever $m, n \to \infty$. We shall denote the space of all $P-$ gai sequences by χ^2. The double sequence x is analytic if there exists a positive number M such that $|x_{jk}|^{\frac{1}{j+k}} < M$ for all j and k. We will denote the set of all analytic double sequences by Λ^2.

Throughout this paper we shall examine our sequence spaces using the following type of transformation:

Definition 2.6. Let $A = \left(a_{k,\ell}^{mn}\right)$ denotes a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k, \ell-$ th term of Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an additional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

Definition 2.7. The four dimensional matrix A is said to be RH-regular if it maps every bounded $P-$gai sequence into a $P-$gai sequence with the same $P-$limit.

In addition to this definition, Robison and Hamilton also presented the following Silverman-Toeplitz type multidimensional characterization of regularity in [39] and [35].

Theorem 2.1. *The four dimensional matrix A is RH-regular if and only if*

$$RH_1 : P - \lim_{k,\ell} a_{k\ell}^{mn} = 0 \text{ for each } m \text{ and } n;$$

$$RH_2 : P - \lim_{k,\ell} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} = 1;$$

$$RH_3 : P - \lim_{k,\ell} \sum_{m=1}^{\infty} |a_{k\ell}^{mn}| = 0 \text{ for each } n;$$

$$RH_4 : P - \lim_{k,\ell} \sum_{n=1}^{\infty} |a_{k\ell}^{mn}| = 0 \text{ for each } m;$$

$$RH_5 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} \text{ is } P- \text{ convergent; and}$$

$$RH_6 : \text{ there exist positive numbers } M \text{ and } N \text{ such that } \sum_{m,n>N} |a_{k\ell}^{mn}| < M.$$

Definition 2.8. A double sequence (x_{mn}) of complex numbers is said to be strongly $A-$ summable to 0, if $P - \lim_{k,\ell} \sum_{m,n} a_{k\ell}^{mn} \left((m+n)! \left|x_{mn} - 0\right|\right)^{\frac{1}{m+n}} = 0$.

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1, 2, 3, \ldots$. A continuous linear functional ϕ on Λ^2 is said to be an invariant mean or a σ-mean if and only if

(1) $\phi(x) \geq 0$ when the sequence $x = (x_{mn})$ has $x_{mn} \geq 0$ for all m, n.

(2) $\phi(e) = 1$ where $e = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$ and

(3) $\phi(\{x_{\sigma(m),\sigma(n)}\}) = \phi(\{x_{mn}\})$ for all $x \in \Lambda^2$.

For certain kinds of mappings σ, every invariant mean ϕ extends the limit functional on the space C of all real convergent sequences in the sense that $\phi(x) = \lim x$ for all $x \in C$ consequently $C \subset V_\sigma$, where V_σ is the set of double analytic sequences all of those $\sigma-$ means are equal.

If $x = (x_{mn})$, set $Tx = (Tx)^{1/m+n} = (x_{\sigma(m),\sigma(n)})$. It can be shown that

$$V_\sigma = \left\{ x \in \Lambda^2 : \lim_{m \to \infty} t_{mn}(x_n)^{1/n} = Le \text{ uniformly in } n,\ L = \sigma - \lim (x_{mn})^{1/m+n} \right\}$$

where

$$t_{mn}(x) = \frac{(x_n + Tx_n + \ldots + T^m x_n)^{1/m+n}}{m+1} \tag{2.1}$$

we say that a double analytic sequence $x = (x_{mn})$ is $\sigma-$ convergent if and only if $x \in V_\sigma$.

Definition 2.9. A double analytic sequence $x = (x_{mn})$ of real numbers is said to be $\sigma-$ convergent to zero provided that

$$P - \lim_{p,q} \frac{1}{pq} \sum_{m=1}^{p} \sum_{n=1}^{q} \left| x_{\sigma^m(k),\sigma^m(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^m(\ell)}} = 0,$$

uniformly in (k, ℓ).

In this case we write $\sigma_2 - \lim x = 0$. We shall also denote the set of all double $\sigma-$ convergent sequences by V_σ^2. Clearly $V_\sigma^2 \subset \Lambda^2$.

One can see that in contrast to the case for single sequences, a $P-$ convergent double sequence need not be $\sigma-$ convergent. But, it is easy to see that every bounded $P-$ convergent double sequence is convergent. In addition, if we let $\sigma(m) = m + 1$, and $\sigma(n) = n + 1$ in then $\sigma-$ convergence of double sequences reduces to the almost convergence of double sequences.

The following definition is a combination of strongly $A-$ summable to zero, modulus function, and $\sigma-$ convergent.

Definition 2.10. Let f be a modulus, $A = (a_{k\ell}^{mn})$ be a nonnegative RH-regular summability matrix method and $e = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}$.

We now define the following sequence spaces:

$$\chi^2(A,f) = \left\{ x \in \chi^2 : P - \lim_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) = 0 \right\},$$

$$\Lambda^2(A,f) = \left\{ x \in \Lambda^2 : \sup_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty \right\}.$$

If $f(x) = x$ then the sequence spaces defined above reduce to the following:

$$\chi^2(A) = \left\{ x \in \chi^2 : P - \lim_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} = 0 \right\},$$

and

$$\Lambda^2(A) = \left\{ x \in \Lambda^2 : \sup_{k\ell} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (a_{k\ell}^{mn}) \left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty \right\}.$$

Some well-known spaces are defined by specializing A and f. For example, if $A = (C,1,1)$, the sequence spaces defined above reduce to $\chi^2(f)$ and $\Lambda^2(f)$ as follows:

$$\chi^2(f) = \left\{ x \in \chi^2 : P - \lim_{k\ell} \frac{1}{k\ell} \sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1} f\left(\left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) = 0 \right\},$$

$$\Lambda^2(f) = \left\{ x \in \Lambda^2 : \sup_{k\ell} \frac{1}{k\ell} \sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty \right\}.$$

As a final illustration, let $A = (C,1,1)$ and $f(x) = x$, we obtain the following spaces:

$$\chi^2 = \left\{ x \in \chi^2 : P - \lim_{k\ell} \frac{1}{k\ell} \sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1} \left((\sigma^m(k) + \sigma^n(\ell))! \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} = 0 \right\}$$

and

$$\Lambda^2 = \left\{ x \in \Lambda^2 : \sup_{k\ell} \frac{1}{k\ell} \sum_{m=0}^{k-1}\sum_{n=0}^{\ell-1} \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty \right\}.$$

3. Main Results

In this section we shall establish some general properties for the above sequence spaces.

Theorem 3.1. *$\chi^2(A,f)$ and $\Lambda^2(A,f)$ are linear spaces over the complex filed $\mathbb{C}$.*

Proof. We shall establish the linearity of $\chi^2(A,f)$ only. The other cases can be treated in a similar manner. Let x and y be elements in $\chi^2(A,f)$. For λ and μ in $\mathbb{C}$ there exist integers M_λ and N_μ such that $|\lambda| < M_\lambda$ and $|\mu| < N_\mu$. From the conditions (2) and (3) of Definition 2.1, we granted the following

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left((\sigma^m(k)+\sigma^n(\ell))!\left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}+\mu y_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}$$
$$\leq M_\lambda \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}$$
$$+N_\mu \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left((\sigma^m(k)+\sigma^n(\ell))!\left|y_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}$$

for all k and ℓ. Since x and y are $\chi^2(A,f)$, we have $\lambda x+\mu y \in \chi^2(A,f)$. Thus $\chi^2(A,f)$ is a linear space. This completes the proof. □

Theorem 3.2. *$\chi^2(A,f)$ is a complete linear topological spaces with the paranorm*

$$g(x)=\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}.$$

Proof. For each $x \in \chi^2(A,f), g(x)$ exists. Clearly $g(\theta)=0$, $g(-x)=g(x)$ and $g(x+y)\leq g(x)+g(y)$. We now show that the scalar multiplication is continuous. Now observe the following

$$g(\lambda x)=\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \leq (1+[\lambda])\, g(x)$$

where $\left[|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}}\right]$ denotes the integer part of $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}}$. In addition observe that $g(x)$ and λ approach to 0 implies $g(\lambda x)$ approaches to 0. For fixed λ, if x approaches to 0 then $g(\lambda x)$ approaches to 0. We now show that fixed $x, g(\lambda x)$ approaches to 0 whenever λ approaches to 0. Let $x \in \chi^2(A,f)$, thus

$$P-\lim_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)=0.$$

If $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}} < 1$ and $M \in \mathbb{N}$ we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\lambda\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)$$
$$\leq \sum_{m\leq M}^{\infty}\sum_{n\leq M}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\lambda\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)$$
$$+\sum_{m\geq M}^{\infty}\sum_{n\geq M}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\lambda\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right).$$

Let $\epsilon > 0$ and choose N such that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2} \tag{3.1}$$

for $k, \ell > N$. Also for each (k, ℓ) with $1 \leq k \leq N$, $1 \leq \ell \leq N$, and since

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \infty,$$

there exists an integer $M_{k,\ell}$ such that

$$\sum_{m>M_{k,\ell}}\sum_{n>M_{k,\ell}}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2}.$$

Taking $M = \inf_{1\leq k\leq N \text{ (or) } 1\leq \ell\leq N}\{M_{k,\ell}\}$, we have for each (k, ℓ) with $1 \leq k \leq N$ or $1 \leq \ell \leq N$

$$\sum_{m>M}\sum_{n>M}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2}.$$

Also from (**??**), for $k, \ell > N$ we have

$$\sum_{m>M}\sum_{n>M}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2}.$$

Thus M is an integer independent of (k, ℓ) such that

$$\sum_{m>M}\sum_{n>M}\left(a_{k\ell}^{mn}\right) f\left(\left((\sigma^m(k)+\sigma^n(\ell))!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2}. \tag{3.2}$$

Further for $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}} < 1$ and for all (k,ℓ)

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\\ &\leq \sum_{m>M}\sum_{n>M}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\\ &+\sum_{m\leq M}\sum_{n\leq M}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right).\end{aligned}$$

For each (k,ℓ) and by the continuity of f as $\lambda \to 0$ we have the following

$$\sum_{m\leq M}\sum_{n\leq M}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right).$$

Now choose $\delta < 1$ such that $|\lambda|^{\frac{1}{\sigma^m(n)+\sigma^n(\ell)}} < \delta$ implies

$$\sum_{m\leq M}\sum_{n\leq M}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \frac{\epsilon}{2}. \tag{3.3}$$

It follows that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|\lambda x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon$$

for all (k,ℓ). Thus $g(\lambda x) \to 0$ as $\lambda \to 0$. Therefore $\chi^2(A,f)$ is a paranormed linear topological space.

Now let us show that $\chi^2(A,f)$ is complete with respect to its paranorm topologies. Let (x_{mn}^s) be a cauchy sequence in $\chi^2(A,f)$. Then, we write $g(x^s - x^t) \to 0$ as $s,t \to \infty$, to mean, as $s,t \to \infty$ for all (k,ℓ)

(3.4)

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|x^s_{\sigma^m(k),\sigma^n(\ell)} - x^t_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) \to 0.$$

Thus for each fixed m and n as $s,t \to \infty$. We are granted

$$f\left((m+n)! \left|x_{mn}^s - x_{mn}^t\right|\right) \to 0$$

and so (x_{mn}^s) is a cauchy sequence in $\mathbb{C}$ for each fixed m and n. Since $\mathbb{C}$ is complete we have $x_{mn}^s \to x_{mn}$, as $s \to \infty$ for each (mn). Now from (2.9), we have for $\epsilon > 0$ there exists a natural number N such that

$$\sum_{m=0}^{\infty}\sum_{n=0 \text{ and } s,t>N}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)! \left|x^s_{\sigma^m(k),\sigma^n(\ell)} - x^t_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right) < \epsilon$$

for (k,ℓ). Since for any fixed natural number M, we have from (2.10)

$$\sum_{m\le M}\sum_{n\le M \text{ and } s,t>N}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x^s_{\sigma^m(k),\sigma^n(\ell)}-x^t_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)<\epsilon$$

for all (k,ℓ), by letting $t\to\infty$ in the above expression we obtain

$$\sum_{m\le M}\sum_{n\le M \text{ and } s>N}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x^s_{\sigma^m(k),\sigma^n(\ell)}-x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)<\epsilon.$$

Since M is arbitrary, by letting $M\to\infty$ we obtain

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x^s_{\sigma^m(k),\sigma^n(\ell)}-x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)<\epsilon$$

for all (k,ℓ). Thus $g\left(x^s-x\right)\to 0$ as $s\to\infty$. Also $\left(x^s\right)$ being a sequence in $\chi^2(A,f)$ by definition of $\chi^2(A,f)$ for each s with

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x^s_{\sigma^m(k),\sigma^n(\ell)}-x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\to 0$$

as $(k,\ell)\to\infty$ thus $x\in\chi^2(A,f)$. This completes the proof. $\square$

Theorem 3.3. *Let $A=\left(a_{k\ell}^{mn}\right)$ be nonnegative matrix such that*

$$\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)<\infty$$

and let f be a modulus, then $\chi^2(A,f)\subset\Lambda^2(A,f)$.

Proof. Let $x\in\chi^2(A,f)$. Then by Definition 2.1 of (2) and (3) of the modulus function we granted the following

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)$$

$$\le\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}-0\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)+f(|0|)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right).$$

There exists an integer N_p such that $|0|\le N_p$. Thus we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right) f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)$$

$$\le\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f\left(\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}-0\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)+N_p f(1)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right).$$

Since $\sup_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)<\infty$ and $x\in\chi^2(A,f)$, we are granted $x\in\Lambda^2(A,f)$ and this completes the proof. $\square$

Theorem 3.4. *Let $A = (a_{k\ell}^{mn})$ be nonnegative matrix such that*

$$\sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_{k\ell}^{mn}) < \infty$$

and let f be a modulus, then $\Lambda^2(A) \subset \Lambda^2(A, f)$.

Proof. Let $x \in \Lambda^2(A)$, so that

$$\sup_{k,\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \infty.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \le t \le \delta$. Consider,

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right)$$
$$= \sum_{m=0}^{\infty} \sum_{n=0 \text{ and } \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \le \delta}^{\infty} a_{k\ell}^{mn} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right)$$
$$+ \sum_{m=0}^{\infty} \sum_{n=0 \text{ and } \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} > \delta}^{\infty} a_{k\ell}^{mn} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right).$$

Then

(3.5)
$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \le \delta}^{\infty} a_{k\ell}^{mn} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \le \epsilon \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn}.$$

For $\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} > \delta$ we use the fact that

$$\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} < \frac{\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta} < \left[1 + \frac{\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta} \right]$$

where $[t]$ denoted the integer part of t and from conditions (2) and (3) of Definition 2.1, modulus function we have

$$f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \le \left[1 + \frac{\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta} \right] f(1)$$
$$\le 2f(1) \frac{\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}}{\delta}.$$

Hence

$$\sum_{m=0}^{\infty} \sum_{n=0 \text{ and } |x_{mn}|^{\frac{1}{m+n}} > \delta}^{\infty} a_{k\ell}^{mn} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right)$$
$$\leq \frac{2f(1)}{\delta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}$$

which together with inequality (3.5) yields the following

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} f\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right)$$
$$\leq \epsilon \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} + \frac{2f(1)}{\delta} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k\ell}^{mn} \left| x_{\sigma^m(k),\sigma^n(\ell)} \right|^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}$$

since $\sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) < \infty$ and $x \in \Lambda^2(A)$ we are granted that $x \in \Lambda^2(A, f)$ and this completes the proof. □

Definition 3.1. Let f be modulus $a_{k\ell}^{mn}-$ a nonnegative RH-regular summability matrix method. Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 < p_{mn} < supp_{mn} = G$ and $D = \max\left(1, 2^{G-1}\right)$. Then for $a_{mn}, b_{mn} \in \mathbb{N}$, the set of complex numbers for all $m, n \in \mathbb{N}$, we have

$$|a_{mn} + b_{mn}|^{\frac{1}{m+n}} \leq D\left\{ |a_{mn}|^{\frac{1}{m+n}} + |b_{mn}|^{\frac{1}{m+n}} \right\}.$$

Let (X, q) be a semi normed space over the field $\mathbb{C}$ of complex numbers with the semi norm q. We define the following sequence spaces:

$$\chi^2(A, f, p, q)$$
$$= x \in \chi^2 : P - \lim_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left(\left(\sigma^m(k) + \sigma^n(\ell) \right)! \left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{p_{mn}}$$
$$= 0.$$

$$\Lambda^2(A, f, p, q)$$
$$= x \in \Lambda^2 : \sup_{k\ell} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(a_{k\ell}^{mn}\right) \left[f\left(q\left(\left| x_{\sigma^m(k),\sigma^n(\ell)} \right| \right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}} \right) \right]^{p_{mn}} < \infty.$$

Theorem 3.5. *Let f_1 and f_2 be two modulus. Then* $\chi^2(A, f_1, p, q) \bigcap \chi^2(A, f_2, p, q) \subseteq \chi^2(A, f_1 + f_2, p, q)$.

Proof. The proof is easy so omitted. □

Remark 3.1. Let f be a modulus q_1 and q_2 be two seminorm on X, we have

(1) $\chi^2(A,f,p,q_1)\bigcap\chi^2(A,f,p,q_2)\subseteq\chi^2(A,f,p,q_1+q_2)$.
(2) If q_1 is stronger than q_2 then $\chi^2(A,f,p,q_1)\subseteq\chi^2(A,f,p,q_2)$.
(3) If q_1 is equivalent to q_2 then $\chi^2(A,f,p,q_1)=\chi^2(A,f,p,q_2)$.

Theorem 3.6. *Let $0\leq p_{mn}\leq r_{mn}$ for all $m,n\in\mathbb{N}$ and let $\left\{\frac{q_{mn}}{p_{mn}}\right\}$ be bounded. Then $\chi^2(A,f,r,q)\subset\chi^2(A,f,p,q)$.*

Proof. Let

$$x\in\chi^2(A,f,r,q)\,, \tag{3.6}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{r_{mn}}.$$

Let

$$t_{mn}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{r_{mn}}, \tag{3.7}$$

we have $\gamma_{mn}=p_{mn}/r_{mn}$. Since $p_{mn}\leq r_{mn}$, we have $0\leq\gamma_{mn}\leq 1$. Let $0<\gamma<\gamma_{mn}$. Then

$$u_{mn}=\begin{cases} t_{mn}, & \text{if } (t_{mn}\geq 1)\\ 0, & \text{if } (t_{mn}<1)\end{cases},$$

$$v_{mn}=\begin{cases} 0, & \text{if } (t_{mn}\geq 1)\\ t_{mn}, & \text{if } (t_{mn}<1)\end{cases}, \tag{3.8}$$

$t_{mn}=u_{mn}+v_{mn}$, $t_{mn}^{\gamma_{mn}}=u_{mn}^{\gamma_{mn}}+v_{mn}^{\gamma_{mn}}$.
Now, it follows that

$$u_{mn}^{\gamma_{mn}}\leq u_{mn}\leq t_{mn},\quad v_{mn}^{\gamma_{mn}}\leq u_{mn}^{\gamma}. \tag{3.9}$$

Since $t_{mn}^{\gamma_{mn}}=u_{mn}^{\gamma_{mn}}+v_{mn}^{\gamma_{mn}}$, we have $t_{mn}^{\gamma_{mn}}\leq t_{mn}+v_{mn}^{\gamma}$. Thus,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)^{r_{mn}}\right]^{\gamma_{mn}}$$
$$\leq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{r_{mn}},$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)^{r_{mn}}\right]^{p_{mn}/r_{mn}}$$
$$\leq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^m(k)+\sigma^n(\ell)\right)!\left|x_{\sigma^m(k),\sigma^n(\ell)}\right|\right)^{\frac{1}{\sigma^m(k)+\sigma^n(\ell)}}\right)\right]^{r_{mn}},$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{p_{mn}}$$
$$\leq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{r_{mn}}.$$

But

$$P-\lim_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{r_{mn}}=0.$$

Therefore we have

$$P-\lim_{k\ell}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{p_{mn}}=0.$$

Hence

$$x\in\chi^{2}\left(A,f,p,q\right). \tag{3.10}$$

From (**??**) and (**??**) we get $x\in\chi^{2}\left(A,f,r,q\right)\subset x\in\chi^{2}\left(A,f,p,q\right)$. □

Theorem 3.7. *The space $\chi^{2}\left(A,f,p,q\right)$ is solid and such are monotones.*

Proof. Let $x=\left(x_{mn}\right)\in\chi^{2}\left(A,f,p,q\right)$ and $\left(\alpha_{mn}\right)$ be a sequence of scalars such that, $\left|\alpha_{mn}\right|\leq 1$ for all $m,n\in\mathbb{N}$. Then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|\alpha_{mn}x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{p_{mn}}$$
$$\leq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{p_{mn}}N,$$
$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|\alpha_{mn}x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{p_{mn}}$$
$$\leq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(a_{k\ell}^{mn}\right)\left[f\left(q\left(\left(\sigma^{m}\left(k\right)+\sigma^{n}\left(\ell\right)\right)!\left|x_{\sigma^{m}(k),\sigma^{n}(\ell)}\right|\right)^{\frac{1}{\sigma^{m}(k)+\sigma^{n}(\ell)}}\right)\right]^{p_{mn}}N,$$

for all $m,n\in\mathbb{N}$. This completes the proof. □

Acknowledgement: The authors would like to thank the referees for their remarks and valuable suggestions that improved the presentation of the paper. The first author also wishes to thank Prof. Dr. F. Başar, Prof. Dr. B. Altay and Prof. Dr. R. Çolak for their valuable moral support in connection with the paper presentation. The third author was supported by Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

References

[1] T. Apostol, *Mathematical Analysis*, Addison-wesley, London, (1978).

[2] M. Basarir and O. Solancan, *On some double sequence spaces*, J. Indian Acad. Math., **21(2)** (1999), 193–200.

[3] C. Bektas and Y. Altin, *The sequence space $\ell_M(p,q,s)$ on seminormed spaces*, *Indian J. Pure Appl. Math.*, **34(4)** (2003), 529–534.

[4] T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, Macmillan and Co.Ltd., New York, (1965).

[5] G. H. Hardy, *On the convergence of certain multiple series*, Proc. Camb. Phil. Soc., **19** (1917), 86–95.

[6] M. A. Krasnoselskii and Y. B. Rutickii, *Convex functions and Orlicz spaces*, Gorningen, Netherlands, (1961).

[7] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., **10** (1971), 379–390.

[8] I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc, **100(1)** (1986), 161–166.

[9] F. Moricz, *Extentions of the spaces c and c_0 from single to double sequences*, Acta. Math. Hung., **57(1-2)**, (1991), 129–136.

[10] F. Moricz and B. E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc., **104**, (1988), 283–294.

[11] M. Mursaleen, M. A. Khan and Qamaruddin, *Difference sequence spaces defined by Orlicz functions*, Demonstratio Math., **Vol. XXXII** (1999), 145–150.

[12] H. Nakano, *Concave modulars*, J. Math. Soc. Japan, **5**(1953), 29–49.

[13] W. Orlicz, Über Raume $\left(L^M\right)$, Bull. Int. Acad. Polon. Sci. A, **(**1936), 93–107.

[14] S. D. Parashar and B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math., **25(4)** (1994), 419–428.

[15] K. Chandrasekhara Rao and N. Subramanian, *The Orlicz space of entire sequences*, Int. J. Math. Math. Sci., **68** (2004), 3755–3764.

[16] W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math., **25**(1973), 973–978.

[17] B. C. Tripathy, *On statistically convergent double sequences*, Tamkang J. Math., **34(3)**, (2003), 231–237.

[18] B. C. Tripathy, M. Et and Y. Altin, *Generalized difference sequence spaces defined by Orlicz function in a locally convex space*, J. Anal. Appl., **1(3)**(2003), 175-192.

[19] A. Turkmenoglu, *Matrix transformation between some classes of double sequences*, J. Inst. Math. Comp. Sci. Math. Ser., **12(1)**, (1999), 23-31.

[20] P. K. Kamthan and M. Gupta, *Sequence spaces and series, Lecture notes, Pure and Applied Mathematics*, 65 Marcel Dekker, In c., New York, **1**981.

[21] A. Gökhan and R. Çolak, *The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$*, Appl. Math. Comput., **157(2)**, (2004), 491-501.

[22] A. Gökhan and R. Çolak, *Double sequence spaces ℓ_2^∞*, ibid., **160(1)**, (2005), 147–153.

[23] M. Zeltser, *Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25*, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, (2001).

[24] M. Mursaleen and O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl., **288(1)**, (2003), 223–231.

[25] M. Mursaleen, *Almost strongly regular matrices and a core theorem for double sequences*, J. Math. Anal. Appl., **293(2)**, (2004), 523–531.

[26] M. Mursaleen and O. H. H. Edely, *Almost convergence and a core theorem for double sequences*, J. Math. Anal. Appl., **293(2)**, (2004), 532–540.

[27] B. Altay and F. Basar, *Some new spaces of double sequences*, J. Math. Anal. Appl., **309(1)**, (2005), 70–90.

[28] F. Başar and Y. Sever, *The space $\mathcal{L}_p$ of double sequences*, Math. J. Okayama Univ, **51**, (2009), 149–157.

[29] N. Subramanian and U. K. Misra, *The semi normed space defined by a double gai sequence of modulus function*, Fasciculi Math., **46**, (2010).

[30] H. Kizmaz, *On certain sequence spaces*, Cand. Math. Bull., **24(2)**, (1981), 169–176.

[31] B. Kuttner, *Note on strong summability*, J. London Math. Soc., **21**(1946), 118–122.

[32] I. J. Maddox, *On strong almost convergence*, Math. Proc. Cambridge Philos. Soc., **85(2)**, (1979), 345–350.

[33] J. Cannor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull., **32(2)**, (1989), 194–198.

[34] A. Pringsheim, *Zurtheorie derzweifach unendlichen zahlenfolgen*, Math. Ann., **53**, (1900), 289–321.

[35] H. J. Hamilton, *Transformations of multiple sequences*, Duke Math. J., **2**, (1936), 29–60.

[36] ———-, *A Generalization of multiple sequences transformation*, Duke Math. J., **4**, (1938), 343–358.

[37] ———-, *Change of Dimension in sequence transformation* , Duke Math. J., **4**, (1938), 341–342.

[38] ———-, *Preservation of partial Limits in Multiple sequence transformations*, Duke Math. J., **4**, (1939), 293–297.

[39] G. M. Robison, *Divergent double sequences and series*, Amer. Math. Soc. Trans., **28**, (1926), 50–73.

[40] L. L. Silverman, *On the definition of the sum of a divergent series*, unpublished thesis, University of Missouri studies, Mathematics series.

[41] O. Toeplitz, *Über allgenmeine linear mittel bridungen*, Prace Matemalyczno Fizyczne (warsaw), **22**, (1911).

[42] F. Başar and B. Atlay, *On the space of sequences of $p-$ bounded variation and related matrix mappings*, Ukrainian Math. J., **55(1)**, (2003), 136–147.

[43] B. Altay and F. Başar, *The fine spectrum and the matrix domain of the difference operator Δ on the sequence space ℓ_p*, $(0 < p < 1)$, Commun. Math. Anal., **2(2)**, (2007), 1–11.

[44] R. Çolak, M. Et and E. Malkowsky, *Some Topics of Sequence Spaces, Lecture Notes in Mathematics*, Firat Univ. Elazig, Turkey, **2004**, pp. 1–63, Firat Univ. Press, (2004), ISBN: 975-394-0386-6.

[1] Department of Mathematics,
SASTRA University,
Thanjavur-613 401, India
E-mail address: nsmaths@yahoo.com

[2] Department of Mathematics and Statistics,
Berhampur University,
Berhampur-760 007, Odissa, India
E-mail address: umakanta_misra@yahoo.com

[3] Faculty of Mathematics and Sciences, University of Niš,
Višegradska 33,18000 Niš,
Serbia
E-mail address: vrakoc@ptt.rs