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SOME NOISELESS CODING THEOREM CONNECTED WITH HAVRDA AND CHARVAT AND TSALLIS'S ENTROPY

SATISH KUMAR [1] AND RAJESH KUMAR [2]

ABSTRACT. A new measure L_{α}^{β}, called average code word length of order α and type β has been defined and its relationship with a result of generalized Havrda and Charvat and Tsallis's entropy has been discussed. Using L_{α}^{β}, some coding theorem for discrete noiseless channel has been proved.

1. INTRODUCTION

Let $\Delta_n = \{P = (p_1, p_2, \ldots, p_N), p_i \geq 0, \sum p_i = 1\}$, $N \geq 2$ be the set of all finite discrete probability distributions, for any probability distribution $(p_1, p_2, \ldots, p_N) = P \in \Delta_n$.
Shannon [23] defined entropy as:

$$H(P) = -\sum p_i \log p_i. \tag{1.1}$$

Throughout this paper, $\sum$ will stand for $\sum_{i=1}^{n}$ unless otherwise stated and logarithms are taken to the base $D\,(D > 1)$.
Let a finite set of N input symbols

$$X = \{x_1, x_2, \ldots, x_N\}$$

be encoded using alphabet of D symbols, then it has been shown by Feinstien [5] that there is uniquely decipherable instantaneous code with length $n_1, n_2, \ldots, n_N$ if and

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only if

$$\sum D^{-n_i} \leq 1 \tag{1.2}$$

where D is the size of code alphabet.
If

$$L = \sum n_i p_i \tag{1.3}$$

is the average codeword length then for a code which satisfies (1.2) it has also been shown by Feinstien [5], that

$$L \geq H(P) \tag{1.4}$$

with equality if and only if

$$n_i = -\log_D p_i \text{ for } \ i = 1, 2, \ldots, N \tag{1.5}$$

and that by suitable encoded into words of long sequences, the average length can be made arbitrary close to $H(P)$. This is Shannon's noiseless coding theorem.

By considering Renyi's [20] entropy, a coding theorem and analogous to the above noiseless coding theorem has been established by Campbell [4] and the authors obtained bounds for it in terms of $H_\alpha(P) = \frac{1}{1-\alpha} \log_D \sum P_I^\alpha$, $\alpha > 0$ $(\alpha \neq 1)$. Kieffer [13] defined a class rules and showed $H_\alpha(P)$ is the best decision rule for deciding which of the two sources can be coded with expected cost of sequences of length n when $n \to \infty$, where the cost of encoding a sequence is assumed to be a function of length only. Further Jelinek [9] showed that coding with respect to Campbell [4] mean length is useful in minimizing the problem of buffer overflow which occurs when the source symbol are being produced at a fixed rate and the code words are stored temporarily in a finite buffer.

Hooda and Bhaker [8] consider the following generalization of Campbell [4] mean length:

$$L^\beta(t) = \frac{1}{t} \log_D \left\{ \frac{\sum p_i^\beta D^{-tn_i}}{\sum p_i^\beta} \right\}, \ \beta \geq 1$$

and proved

$$H_\alpha^\beta\ (P)\ \leq L^\beta(t) <\ H_\alpha^\beta\ (P) + 1, \quad \alpha > 0,\ \alpha \neq 1,\ \beta \geq 1$$

under the condition

$$\sum p_i^{\beta-1} D^{-n_i} \leq \sum p_i^\beta$$

where $H_\alpha^\beta\,(P)$ is generalized entropy of order $\alpha = \frac{1}{1+t}$ and type β studied by Aczel and Daroczy [1] and Kapur [10]. It may be seen that the mean codeword length (1.3) had been generalized parametrically and their bounds had been studied in terms of generalized measures of entropies. Here we give another generalization of (1.3) and study its bounds in terms of generalized entropy of orderα and type β.

Longo [15], Gurdial and Pessoa [6], Singh, Kumar and Tuteja [24], Parkash and Sharma [18], Hooda and Bhaker [8], Khan, Bhat and Pirzada [12], Arndt [2], Baig and Ahmad [3], Kerridge [11], Kraft [14], Mc-Millan [16], Pirzada and Bhat [19], Roy [21] and Satish Kumar [22] have studied generalized coding theorems by considering different generalized measure of (1.1) and (1.3) under condition (1.2) of unique decipherability.

In this paper we study some coding theorems by considering a new function depending on parameters α and β. Our motivation for studying this new function is that it generalizes some entropy function already existing in literature Havrda and Charvat [7] and Tsallis [25] entropy which is used in physics.

2. Coding Theorem

In this section, we define information measure as

$$H_\alpha^\beta\,(P) = \frac{1}{\alpha-1}\left[1 - \frac{\sum p_i^{\alpha\beta}}{\sum p_i^\beta}\right], \tag{2.1}$$

where $\alpha > 0\,(\neq 1)$, $\beta > 0$, $p_i > 0$, $\sum p_i = 1$, $i = 1, 2, \ldots, N$.
(i) When $\beta = 1$, (2.1) reduces to Havrda and Charvat [7] and Tsallis's [25] entropy i.e.,

$$H_\alpha\,(P) = \frac{1}{\alpha-1}\left[1 - \sum p_i^\alpha\right]. \tag{2.2}$$

(ii) When $\beta = 1$, $\alpha \to 1$ then (2.1) reduces to Shannon's [23] entropy

$$H\,(P) = -\sum p_i \log p_i. \tag{2.3}$$

(iii) When $\alpha \to 1$ then (2.1) reduces to Mathur and Mitter's [17] entropy for the β-power distribution, i.e.,

$$H^\beta\,(P) = -\frac{\sum p_i^\beta \log p_i^\beta}{\sum p_i^\beta}. \tag{2.4}$$

Definition 2.1. The mean length L_α^β with respect to information measure is defined as

$$L_\alpha^\beta = \frac{1}{\alpha-1}\left[1-\left\{\sum p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right\}^\alpha\right], \tag{2.5}$$

where $\alpha > 0\,(\neq 1)$, $\beta > 0$, $p_i > 0$, $\sum p_i = 1$, $i = 1, 2, \ldots, N$.

(i) When $\beta = 1$, Then (2.5) reduces to new mean codeword length, i.e.,

$$L_\alpha = \frac{1}{\alpha-1}\left[1-\left\{\sum p_i D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right\}^\alpha\right]. \tag{2.6}$$

(ii) When $\beta = 1$, $\alpha \to 1$, then (2.5) reduces to mean code length defined by Shannon [23], i.e.,

$$L = \sum n_i p_i.$$

We establish a result, that in a sense, provides a characterization of $H_\alpha^\beta(P)$ under the condition of unique decipherability.

Theorem 2.1. *For all integers $D > 1$*

$$L_\alpha^\beta \geq H_\alpha^\beta(P) \tag{2.7}$$

under the condition (1.2) *equality holds if and only if*

$$n_i = -\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right). \tag{2.8}$$

Proof. We use Holder's inequality

$$\sum x_i y_i \geq \left(\sum x_i^p\right)^{\frac{1}{p}} \left(\sum y_i^q\right)^{\frac{1}{q}} \tag{2.9}$$

for all $x_i \geq 0$, $y_i \geq 0$, $i = 1, 2, \ldots, N$ when $P < 1$ ($\neq$1) and $p^{-1} + q^{-1} = 1$, with equality if and only if there exists a positive number c such that

$$x_i^p = c y_i^q. \tag{2.10}$$

Setting

$$x_i = p_i^{\frac{\alpha\beta}{\alpha-1}} \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha-1}} D^{-n_i},$$

$$y_i = p_i^{\frac{\alpha\beta}{1-\alpha}} \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{1-\alpha}},$$

$p = \frac{\alpha-1}{\alpha}$ and $q = 1-\alpha$ in (2.9) and using (1.2) we obtain the result (2.7) after simplification for $\frac{1}{\alpha-1} > 0$ as $\alpha > 1$.

The equality holds if and only if $D^{-n_i} = \frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}$, $i = 1, 2, \ldots, N$ which is equivalent to

$$n_i = -\log_D(\frac{p_i^{\alpha\beta}}{\Sigma p_i^{\alpha\beta}}), \quad i = 1, 2, \ldots, N.$$

□

Theorem 2.2. *For every code with lengths* $\{n_i\}$, $i = 1, 2, \ldots, N$, L_α^β *can be made to satisfy*

$$L_\alpha^\beta < H_\alpha^\beta(P)\, D^{1-\alpha} + \frac{1}{\alpha - 1}\left[1 - D^{1-\alpha}\right]. \tag{2.11}$$

Proof. Let n_i be the positive integer satisfying, the inequalities

$$-\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) \le n_i < -\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) + 1. \tag{2.12}$$

Consider the intervals

$$\delta_i = \left[-\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right), -\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) + 1\right] \tag{2.13}$$

of length 1. In every δ_i, there lies exactly one positive number n_i such that

$$0 < -\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) \le n_i < -\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) + 1. \tag{2.14}$$

It can be shown that the sequence $\{n_i\}$, $i = 1, 2, \ldots, N$ thus defined, satisfies (1.2). From (2.14) we have

$$n_i < -\log_D\left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) + 1$$

$$\Rightarrow \quad D^{-n_i} > \left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right) D^{-1}$$

$$\Rightarrow \quad D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} > \left(\frac{p_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right)^{\frac{\alpha-1}{\alpha}} D^{\frac{1-\alpha}{\alpha}}, \tag{2.15}$$

multiplying both sides of (2.15) by $p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}}$, summing over $i = 1, 2, \ldots, N$ and simplification for $\frac{1}{\alpha-1}$ as $\alpha > 1$, gives (2.11). □

Theorem 2.3. *For every code with length* $\{n_i\}$, $i = 1, 2, \ldots, N$ *of Theorem 2.1,* L_α^β *can be made to satisfy*

$$L_\alpha^\beta \ge H_\alpha^\beta(P) > H_\alpha^\beta(P)\, D + \frac{1}{\alpha - 1}(1 - D). \tag{2.16}$$

Proof. Suppose

$$\bar{n}_i = -\log_D\left(\frac{P_i^{\alpha\beta}}{\sum p_i^{\alpha\beta}}\right). \tag{2.17}$$

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy 'equality' in Holder's inequality (2.9). Moreover, $\bar{n}_i$ satisfies Kraft's inequality (1.2). Suppose n_i is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, then obviously, n_i satisfies (1.2).

Since $\alpha > 0\ (\neq 1)$, we have

$$\begin{aligned}\left(\sum p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-n_i\frac{(\alpha-1)}{\alpha}}\right)^\alpha &\leq \left(\sum p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-\bar{n}_i\frac{(\alpha-1)}{\alpha}}\right)^\alpha \\ &< D\left(\sum p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-\bar{n}_i\frac{(\alpha-1)}{\alpha}}\right)^\alpha.\end{aligned} \tag{2.18}$$

Hence, since

$$\left(\sum p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-\bar{n}_i\frac{(\alpha-1)}{\alpha}}\right)^\alpha = \frac{\sum p_i^{\alpha\beta}}{\sum p_i^\beta},$$

(2.18) becomes

$$\left(\sum p_i^\beta \left(\frac{1}{\sum p_i^\beta}\right)^{\frac{1}{\alpha}} D^{-n_i\frac{(\alpha-1)}{\alpha}}\right)^\alpha \leq \frac{\sum p_i^{\alpha\beta}}{\sum p_i^\beta} < D\left(\frac{\sum p_i^{\alpha\beta}}{\sum p_i^\beta}\right)$$

which gives (2.16). □

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[1] Department of Mathematics,
G.I.M.T. (Kanipla) Kurukshetra,
Haryana, India
E-mail address: drsatish74@rediffmail.com

[2] Department of Mathematics, Hindu College,
University of Delhi,
Delhi-7, India
E-mail address: rajeshhctm@rediffmail.com