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THE IMPROVEMENT OF THE VALUE DISTRIBUTION ON $f + a(f')^n$

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Abstract. Let $f(z)$ be a transcendental meromorphic function in the plane and let $a(\neq 0)$, b be two finite complex numbers. Then for positive integer $n \geq 3$, we have $(n-1)T(r, f') \leq 3\overline{N}(r, f) + 4\overline{N}(r, \frac{1}{f+a(f')^n-b}) + S(r, f)$.

1. Introduction and main results

Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. We will use the standard notations of Nevanlinna's value distribution theory such that $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $m(r, f)$ and so on, as found in [1].

It is interesting to combine the function and it's derivative. In 1959, Hayman prove the following theorem.

Theorem A. [2] *Let $f(z)$ be transcendental meromorphic function in the plane, a a finite non-zero complex number and let $n \geq 5$ be a positive integer. Then $f' + af^n$ assumes every finite complex number infinitely often.*

In 1979, Mues [3] show that for case $n = 3$ or 4, Theorem A is not right.

In 1994, Ye Yasheng studied the value distribution of $f + a(f')^n$ which is similar to Theorem A,and get the following results.

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Theorem B. [4] *Let $f(z)$ be a transcendental meromorphic function in the plane and let $a(\neq 0)$, b be two finite complex numbers. Then for positive integer $n \geq 3$, we have*

$$(n-1)T(r,f') \leq 4\overline{N}(r,f) + 9N(r,\frac{1}{f+a(f')^n-b}) + S(r,f).$$

In 2008, M. L. Fang and Lawrence Zaclman improved Theorem B.

Theorem C. [5] *Let $f(z)$ be a transcendental meromorphic function in the plane and let $a(\neq 0)$, b be two finite complex numbers. Then for positive integer $n \geq 3$, we have*

$$(n-1)T(r,f') \leq 3\overline{N}(r,f) + 4N(r,\frac{1}{f+a(f')^n-b}) + S(r,f').$$

There is a natural question: "Can we replace $N(r,\frac{1}{f+a(f')^n-b})$ by $\overline{N}(r,\frac{1}{f+a(f')^n-b})$ in Theorem C?" In this paper, we will do this work and get a stronger inequality as following.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function in the plane and let $a(\neq 0)$, b be two finite complex numbers. Then for positive integer $n \geq 3$, we have*

$$(n-1)T(r,f') \leq 3\overline{N}(r,f) + 4\overline{N}(r,\frac{1}{f+a(f')^n-b}) + S(r,f).$$

2. Some Lemmas

Lemma 2.1. [6] *Let $f(z)$ be a meromorphic function in the plane. For positive integer k, $f^{((k+1))} \not\equiv 0$. Then*

$$m(r,\frac{f^{(k)}}{f}) = S(r,f^{(k)}).$$

Before we give Lemma 2.2, we first define a differential polynomial. A differential polynomial P of f is defined by

$$P(z) = \sum_{t=1}^{n} \phi_t(z) \tag{2.1}$$

where

$$\phi_t(z) = \alpha_t(z) \prod_{j=0}^{k} (f^{(j)}(z))^{S_{tj}},$$

$\alpha_t \not\equiv 0$, the S_{tj} are non-negative integers and $T(r, \alpha_t) = S(r, f)$ for all t. Let

$$\overline{d}(P) = \max_{1 \le t \le n} \sum_{j=0}^{k} S_{tj} \quad \text{and} \quad \underline{d}(P) = \min_{1 \le t \le n} \sum_{j=0}^{k} S_{tj}.$$

Then $\overline{d}(P)$ is the degree of P, while $\underline{d}(P)$ is the minimal degree of the constituent differential monomials. If $\overline{d}(P) = \underline{d}(P)$, P is said to be homogeneous, and inhomogeneous otherwise.

Lemma 2.2. [7] *Let $f(z)$ be a transcendental meromorphic function in the plane, P a nonconstant differential polynomials in f such that $\underline{d}(P) \ge 2$ and let*

$$Q = \max_{1 \le t \le n} \{ \sum_{j=1}^{k} j S_{tj} \}.$$

Then

$$\underline{d}(P) T(r, f) \le (Q + 1) \overline{N}(r, \frac{1}{f}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{P - 1}) + S(r, f).$$

Proof. In fact the conclusion of Lemma 2.2 is the not the last conclusion of [7]. At the last part of proof of Theorem 1 from [7], if we omit the inequality of $\overline{N}(r, f) \le T(r, f)$ and remain the term $\overline{N}(r, f)$, then we can get the conclusion of Lemma 2.2. □

Lemma 2.3. *Let $f(z)$ be a transcendental meromorphic function in the plane and let $a (\ne 0)$ be a finite number. Then*

$$(n + 1) T(r, f) \le 2 \overline{N}(r, \frac{1}{f}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f^n f' - a}) + S(r, f).$$

Proof. In fact, the conclusion of Lemma 2.3 had been proved by S.Z.Ye ([8]). But now we give a shorter proof. From Nevanlinna Theory, we know that '1' in the term of $P - 1$ of Lemma 2.2 is not essential. It can be replace by any finite number $a (\ne 0)$. Hence, let $P = f^n f'$. In this case, $Q = 1$ and $\underline{d}(P) = n + 1$. Then we can get Lemma 2.3 immediately from Lemma 2.2. □

3. Proof of Theorem

Proof. Let

$$g = f + a(f')^n - b, \quad \phi = \frac{g'}{g}. \tag{3.1}$$

If $\phi \equiv 0$, then $g' \equiv 0$, that is to say

$$f'[1 + na(f')^{n-2} f''] \equiv 0.$$

From the equality above, we can obtain that f is entire function in $\mathbb{C}$. If there exists z_0 such that $f'(z_0) \neq 0$, then there also exists $D_\delta(z_0)$ (which is a neighborhood of z_0 with radius $\delta > 0$) such that $f'(z) \neq 0$ in $D_\delta(z_0)$. Then from the equality we can get $1 + na(f')^{n-2}f'' \equiv 0$ in $D_\delta(z_0)$. Hence by Uniqueness Theorem, we can obtain $1 + na(f')^{n-2}f'' \equiv 0$ in the plane $\mathbb{C}$.

If there exists z_0 such that $1+na(f'(z_0))^{n-2}f''(z_0) \neq 0$, then there also exists $D_\delta(z_0)$ (which is a neighborhood of z_0 with radius $\delta > 0$) such that $1+na(f'(z_0))^{n-2}f''(z_0) \neq 0$ in $D_\delta(z_0)$. Then from the equality we can get $f'(z) \equiv 0$ in $D_\delta(z_0)$. Hence by Uniqueness Theorem, we can obtain $f'(z) \equiv 0$ in the plane $\mathbb{C}$.

Then $f'(z) \equiv 0$ or $1 + na(f')^{n-2}f'' \equiv 0$. Hence f is constant or a polynomial with degree 2, a contradiction with f is transcendental. Hence $\phi \not\equiv 0$.

From Nevanlinna theory and (3.1), we can obtain $T(r, g') \leq O(T(r, f'))$. Together with Lemma 2.1, we can get

$$m(r, \phi) = S(r, f'). \tag{3.2}$$

From (3.1), we can obtain

$$f'[1 + na(f')^{n-2}f''] = \phi[f + a(f')^n - b]. \tag{3.3}$$

$f'(z_0) = 0 \Rightarrow [1 + na(f')^{n-2}f''](z_0) = 1 \neq 0$ and $[1 + na(f')^{n-2}f''](z_0) = 0 \Rightarrow f'(z_0) \neq 0$ (otherwise $f'(z_0) = 0$, then $[1 + na(f')^{n-2}f''](z_0) = 1$, a contradiction). Hence zeros of f is different with all zeros of $1 + na(f')^{n-2}f''$. And from (3.3) we can get that the zeros of f and $1+na(f')^{n-2}f''$ is either from the zeros of ϕ or $f+a(f')^n-b$.

By Nevanlinna First Fundamental Theory and (3.3) and (3.2), we can get

$$\begin{aligned}
\overline{N}(r, \frac{1}{f'}) + \overline{N}(r, \frac{1}{(f')^{n-2}f'' + \frac{1}{na}}) &\leq \overline{N}(r, \frac{1}{\phi}) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) \\
&\leq N(r, \frac{1}{\phi}) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) \\
&\leq T(r, \phi) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) + S(r, f') \\
&= N(r, \phi) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) + S(r, f').
\end{aligned} \tag{3.4}$$

Since $g = f + a(f')^n - b$, $\phi = \frac{g'}{g}$, it is easy to see that the poles of ϕ are either from the pole of g' or the zeros of g, and the multiplicity of the pole of ϕ is simple. Hence

$$\begin{aligned} N(r,\phi) &\le \overline{N}(r,g') + \overline{N}(r,\frac{1}{g}) = \overline{N}(r,g) + \overline{N}(r,\frac{1}{g}) \\ &= \overline{N}(r,f) + \overline{N}(r,\frac{1}{f+a(f')^n-b}). \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we can get

$$\overline{N}(r,\frac{1}{f'}) + \overline{N}(r,\frac{1}{(f')^{n-2}f'' + \frac{1}{na}}) \le \overline{N}(r,f) + 2\overline{N}(r,\frac{1}{f+a(f')^n-b}) + S(r,f'). \tag{3.6}$$

Let $g = f'$ and $m = n-2$. By Lemma 2.3 we can get

$$(m+1)T(r,g) \le 2\overline{N}(r,\frac{1}{g}) + \overline{N}(r,g) + \overline{N}(r,\frac{1}{(g)^{n-2}g' - \frac{1}{na}}) + S(r,g). \tag{3.7}$$

By putting $g = f'$ and $m = n-2$ into (3.7), we can get

$$(n-1)T(r,f') \le 2\overline{N}(r,\frac{1}{f'}) + \overline{N}(r,f') + \overline{N}(r,\frac{1}{(f')^{n-2}f'' - \frac{1}{na}}) + S(r,f'). \tag{3.8}$$

Together with (3.6) and (3.8), we have

$$\begin{aligned} (n-1)T(r,f') &\le 2\overline{N}(r,\frac{1}{f'}) + \overline{N}(r,f') + \overline{N}(r,\frac{1}{(f')^{n-2}f'' - \frac{1}{na}}) + S(r,f') \\ &\le 2[\overline{N}(r,\frac{1}{f'}) + \overline{N}(r,\frac{1}{(f')^{n-2}f'' - \frac{1}{na}})] + \overline{N}(r,f') + S(r,f') \\ &= 2[\overline{N}(r,\frac{1}{f'}) + \overline{N}(r,\frac{1}{(f')^{n-2}f'' - \frac{1}{na}})] + \overline{N}(r,f) + S(r,f') \\ &\le 2[\overline{N}(r,f) + 2\overline{N}(r,\frac{1}{f+a(f')^n-b})] + \overline{N}(r,f) + S(r,f') \\ &= 3\overline{N}(r,f) + 4\overline{N}(r,\frac{1}{f+a(f')^n-b}) + S(r,f'). \end{aligned}$$

Hence, Theorem 1.1 has been proved completely. □

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