THE IMPROVEMENT OF THE VALUE DISTRIBUTION ON
\[ f + a(f')^n \]

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Abstract. Let \( f(z) \) be a transcendental meromorphic function in the plane and let \( a(\neq 0), b \) be two finite complex numbers. Then for positive integer \( n \geq 3 \), we have \((n - 1)T(r, f') \leq 3N(r, f) + 4N(r, \frac{1}{f + a(f')^n - b}) + S(r, f)\).

1. Introduction and main results

Let \( f \) be a nonconstant meromorphic function in the whole complex plane \( \mathbb{C} \). We will use the standard notations of Nevanlinna’s value distribution theory such that \( T(r, f), N(r, f), N(r, f), m(r, f) \) and so on, as found in [1].

It is interesting to combine the function and it’s derivative. In 1959, Hayman prove the following theorem.

**Theorem A.** [2] Let \( f(z) \) be transcendental meromorphic function in the plane, a finite non-zero complex number and let \( n \geq 5 \) be a positive integer. Then \( f' + af^n \) assumes every finite complex number infinitely often.

In 1979, Mues [3] show that for case \( n = 3 \) or 4, Theorem A is not right.

In 1994, Ye Yasheng studied the value distribution of \( f + a(f')^n \) which is similar to Theorem A, and get the following results.

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**Theorem B.** [4] Let \( f(z) \) be a transcendental meromorphic function in the plane and let \( a(\neq 0), b \) be two finite complex numbers. Then for positive integer \( n \geq 3 \), we have

\[
(n - 1)T(r, f') \leq 4N(r, f) + 9N(r, \frac{1}{f + a(f)^n - b}) + S(r, f).
\]

In 2008, M. L. Fang and Lawrence Zaclman improved Theorem B.

**Theorem C.** [5] Let \( f(z) \) be a transcendental meromorphic function in the plane and let \( a(\neq 0), b \) be two finite complex numbers. Then for positive integer \( n \geq 3 \), we have

\[
(n - 1)T(r, f') \leq 3N(r, f) + 4N(r, \frac{1}{f + a(f)^n - b}) + S(r, f').
\]

There is a natural question: "Can we replace \( N(r, \frac{1}{f + a(f)^n - b}) \) by \( N(r, \frac{1}{f + a(f)^n - b}) \) in Theorem C?" In this paper, we will do this work and get a stronger inequality as following.

**Theorem 1.1.** Let \( f(z) \) be a transcendental meromorphic function in the plane and let \( a(\neq 0), b \) be two finite complex numbers. Then for positive integer \( n \geq 3 \), we have

\[
(n - 1)T(r, f') \leq 3N(r, f) + 4N(r, \frac{1}{f + a(f)^n - b}) + S(r, f).
\]

2. **Some Lemmas**

**Lemma 2.1.** [6] Let \( f(z) \) be a meromorphic function in the plane. For positive integer \( k, f^{((k+1))} \neq 0 \). Then

\[
m(r, \frac{f^{(k)}}{f}) = S(r, f^{(k)}).
\]

Before we give Lemma 2.2, we first define a differential polynomial. A differential polynomial \( P \) of \( f \) is defined by

\[
P(z) = \sum_{t=1}^{n} \phi_t(z)
\]

where

\[
\phi_t(z) = \alpha_t(z) \prod_{j=0}^{k} (f^{(j)}(z))^{S_{tj}},
\]
\( \alpha_t \neq 0 \), the \( S_{ij} \) are non-negative integers and \( T(r, \alpha_t) = S(r, f) \) for all \( t \). Let

\[
\overline{d}(P) = \max_{1 \leq t \leq n} \sum_{j=0}^{k} S_{tj} \quad \text{and} \quad \underline{d}(P) = \min_{1 \leq t \leq n} \sum_{j=0}^{k} S_{tj}.
\]

Then \( \overline{d}(P) \) is the degree of \( P \), while \( \underline{d}(P) \) is the minimal degree of the constituent differential monomials. If \( \underline{d}(P) = \overline{d}(P) \), \( P \) is said to be homogeneous, and inhomogeneous otherwise.

Lemma 2.2. [7] Let \( f(z) \) be a transcendental meromorphic function in the plane, \( P \) a nonconstant differential polynomials in \( f \) such that \( \overline{d}(P) \geq 2 \) and let

\[
Q = \max_{1 \leq t \leq n} \left\{ \sum_{j=1}^{k} jS_{tj} \right\}.
\]

Then

\[
d(P)T(r, f) \leq (Q + 1)N(r, \frac{1}{f}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{P - 1}) + S(r, f).
\]

Proof. In fact the conclusion of Lemma 2.2 is not the last conclusion of [7]. At the last part of proof of Theorem 1 from [7], if we omit the inequality of \( \overline{N}(r, f) \leq T(r, f) \) and remain the term \( \overline{N}(r, f) \), then we can get the conclusion of Lemma 2.2. \( \Box \)

Lemma 2.3. Let \( f(z) \) be a transcendental meromorphic function in the plane and let \( a(\neq 0) \) be a finite number. Then

\[
(n + 1)T(r, f) \leq 2\overline{N}(r, \frac{1}{f}) + \overline{N}(r, f) + \overline{N}(r, \frac{1}{f^{n-1} - a}) + S(r, f).
\]

Proof. In fact, the conclusion of Lemma 2.3 had been proved by S.Z.Ye ([8]). But now we give a shorter proof. From Nevanlinna Theory, we know that ‘1’ in the term of \( P - 1 \) of Lemma 2.2 is not essential. It can be replace by any finite number \( a(\neq 0) \). Hence, let \( P = f^n f' \). In this case, \( Q = 1 \) and \( \underline{d}(P) = n + 1 \). Then we can get Lemma 2.3 immediately from Lemma 2.2. \( \Box \)

3. Proof of Theorem

Proof. Let

\[
g = f + a(f')^n - b, \quad \phi = \frac{g'}{g}.
\]

If \( \phi \equiv 0 \), then \( g' \equiv 0 \), that is to say

\[
f'[1 + na(f')^{n-2} f''] \equiv 0.
\]
From the equality above, we can obtain that \( f \) is entire function in \( \mathbb{C} \). If there exists \( z_0 \) such that \( f'(z_0) \neq 0 \), then there also exists \( D_\delta(z_0) \) (which is a neighborhood of \( z_0 \) with radius \( \delta > 0 \)) such that \( f'(z) \neq 0 \) in \( D_\delta(z_0) \). Then from the equality we can get \( 1 + na(f')^{n-2}f'' \equiv 0 \) in \( D_\delta(z_0) \). Hence by Uniqueness Theorem, we can obtain 
\[
1 + na(f')^{n-2}f'' \equiv 0 \text{ in the plane } \mathbb{C}.
\]

If there exists \( z_0 \) such that \( 1 + na(f'(z_0))^{n-2}f''(z_0) \neq 0 \), then there also exists \( D_\delta(z_0) \) (which is a neighborhood of \( z_0 \) with radius \( \delta > 0 \)) such that \( 1 + na(f'(z_0))^{n-2}f''(z_0) \neq 0 \) in \( D_\delta(z_0) \). Then from the equality we can get \( f'(z) \equiv 0 \) in \( D_\delta(z_0) \). Hence by Uniqueness Theorem, we can obtain \( f'(z) \equiv 0 \) in the plane \( \mathbb{C} \).

Then \( f'(z) \equiv 0 \) or \( 1 + na(f')^{n-2}f'' \equiv 0 \). Hence \( f \) is constant or a polynomial with degree 2, a contradiction with \( f \) is transcendental. Hence \( \phi \neq 0 \).

From Nevanlinna theory and (3.1), we can obtain \( T(r, g') \leq O(T(r, f')) \). Together with Lemma 2.1, we can get
\[
(3.2) \quad m(r, \phi) = S(r, f').
\]

From (3.1), we can obtain
\[
(3.3) \quad f'[1 + na(f')^{n-2}f''] = \phi[f + a(f')^n - b].
\]

\( f'(z_0) = 0 \Rightarrow [1 + na(f')^{n-2}f''](z_0) = 1 \neq 0 \) and \( [1 + na(f')^{n-2}f''](z_0) = 0 \Rightarrow f'(z_0) \neq 0 \) (otherwise \( f'(z_0) = 0 \), then \( [1 + na(f')^{n-2}f''](z_0) = 1 \), a contradiction).

Hence zeros of \( f \) is different with all zeros of \( 1 + na(f')^{n-2}f'' \). And from (3.3) we can get that the zeros of \( f \) and \( 1 + na(f')^{n-2}f'' \) is either from the zeros of \( \phi \) or \( f + a(f')^n - b \).

By Nevanlinna First Fundamental Theory and (3.3) and (3.2), we can get
\[
\begin{align*}
\overline{N}(r, \frac{1}{f'}) + \overline{N}(r, \frac{1}{(f')^{n-2}f'' + \frac{1}{na}}) & \leq \overline{N}(r, \frac{1}{\phi}) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) \\
& \leq N(r, \frac{1}{\phi}) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) \\
& \leq T(r, \phi) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) + S(r, f') \\
& = N(r, \phi) + \overline{N}(r, \frac{1}{f + a(f')^n - b}) + S(r, f').
\end{align*}
\]

(3.4)
Since \( g = f + a(f')^n - b \), \( \phi = \frac{f'}{g} \), it is easy to see that the poles of \( \phi \) are either from the pole of \( g' \) or the zeros of \( g \), and the multiplicity of the pole of \( \phi \) is simple. Hence

\[
N(r, \phi) \leq \overline{N}(r, g') + \overline{N}(r, \frac{1}{g}) = \overline{N}(r, g) + \overline{N}(r, \frac{1}{g}) = \overline{N}(r, f) + \overline{N}(r, \frac{1}{f + a(f')^n - b})).
\]

(3.5)

From (3.4) and (3.5), we can get

\[
N(r, \frac{1}{f'}) + \overline{N}(r, \frac{1}{(f')^n f'' - \frac{1}{na}}) \leq 2\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f + a(f')^n - b}) + S(r, f').
\]

(3.6)

Let \( g = f' \) and \( m = n - 2 \). By Lemma 2.3 we can get

\[
(m + 1)T(r, g) \leq 2\overline{N}(r, \frac{1}{g}) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{(g)^n - 2g' - \frac{1}{na}}) + S(r, g).
\]

(3.7)

By putting \( g = f' \) and \( m = n - 2 \) into (3.7), we can get

\[
(n - 1)T(r, f') \leq 2\overline{N}(r, \frac{1}{f'}) + \overline{N}(r, f') + \overline{N}(r, \frac{1}{(f')^n - 2f'' - \frac{1}{na}}) + S(r, f').
\]

(3.8)

Together with (3.6) and (3.8), we have

\[
(n - 1)T(r, f') \leq 2\overline{N}(r, \frac{1}{f'}) + \overline{N}(r, f') + \overline{N}(r, \frac{1}{(f')^n - 2f'' - \frac{1}{na}}) + S(r, f')
\]

\[
\leq 2[\overline{N}(r, \frac{1}{f'}) + \overline{N}(r, \frac{1}{(f')^n - 2f'' - \frac{1}{na}})] + \overline{N}(r, f') + S(r, f')
\]

\[
= 2[\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{(f')^n - 2f'' - \frac{1}{na}})] + \overline{N}(r, f) + S(r, f')
\]

\[
\leq 2[\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f + a(f')^n - b})] + \overline{N}(r, f) + S(r, f')
\]

\[
= 3\overline{N}(r, f) + 4\overline{N}(r, \frac{1}{f + a(f')^n - b}) + S(r, f').
\]

Hence, Theorem 1.1 has been proved completely. \( \square \)

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References


