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ON A BESSACK'S INEQUALITY RELATED TO OPIAL'S AND HARDY'S

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Abstract. Bessack [2] in 1979 used Holder's inequality to obtain an integral inequality which has as special cases Opial's and Hardy's. Here, using mainly Jensen's inequality for convex functions, with a non-negative, non-decreasing function in the operator, we obtain an integral inequality which is similar to Bessaack's but now containing a refinement term. When $l'(x)$ in Bessack [2] and f in Imoru and Adeagbo-Sheikh [4] are restricted to being non-decreasing, these two inequalities become special cases of our results.

1. Introduction and preliminaries

In 1979 Beesack [2], using only the Holder's inequality, gave conditions under which the weighted inequality

$$\left(\int_a^b |l(x)|^q |l^{'}(x)|^s v(x)dx\right)^{1/q+s} \leq c\left(\int_a^b |l^{'}(t)|^p u(t)dt\right)^{1/p} \tag{1.1}$$

holds for all l with $l(a) = 0$. Where $p > 1$, $q > 0$ and $0 < s < p$. The case of negative values of p, q and s was also considered.

The inequality (1.1) has as special cases the Opial's inequality

$$\int_0^h |l(x)l'(x)|dx \leq \frac{h}{2}\int_0^h |l'(x)|^2 dx, \tag{1.2}$$

and the weighted Hardy's inequality

$$\left(\int_a^b |\int_a^x f(t)dt|^q v(x)dx\right)^{1/q} \leq C\left(\int_a^b |f(t)|^p u(t)dt\right)^{1/p}. \tag{1.3}$$

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For, if $q = s = 1$, $u(t) = v(x) = 1$ and $p = 2$, inequality (1.1) is equivalent to Inequality (1.2) and if $s = 0$, Inequality (1.1) is equivalent to Inequality (1.3).
For the proof of Inequality (1.2) see [6] and for the proof of Inequality (1.3) see [5].

In 2007 Imoru and Adeagbo- Sheikh [4], established the following result:
Suppose g is continuous, non-decreasing on $[a, b]$, $0 \le a \le b < \infty$, with $g(x) > 0$ for $x > 0$, $1 \le p \le q < \infty$ and f is non-negative Lebesgue-Stieltjes integrable with respect to g on $[a, b]$. Then

$$\begin{aligned}(1.4)\quad &\left[\int_a^b g(x)^{\frac{\delta q}{p}-1}\left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{q}{p}(1-p)}\left[\int_x^b f(t)dg(t)\right]^q dg(x) + A_2(p,q,a,b,\delta)\right]^{\frac{1}{q}} \\ &\qquad \le C_2(p,q,\delta)\left[\int_a^b g(x)^{\delta p-1}\left[g(x)f(x)\right]^p dg(x)\right]^{\frac{1}{p}},\end{aligned}$$

where

$$A_2(p,q,a,b,\delta) = \frac{p}{q}(\delta)^{\frac{q}{p}(1-p)-1} g(a)^{\frac{\delta q}{p}}\theta_b(a)^{\frac{q}{p}}, \qquad C_2(p,q,\delta) = \left[\frac{p}{q}\delta^{\frac{q}{p}(1-p)-1}\right]^{\frac{1}{q}} \quad \delta > 0.$$

A recent trend in inequalities, is to establish, mainly by Jensen's inequality (see [1], [3], [4] and [7]) and its generalization due to Steffensen, some general inequalities that include as special cases that are of independent interest and which were originally proved by quite different methods.

The purpose of this paper is to use mainly Jensen's inequality to obtain an integral inequality which is related to both Opial's and Hardy's.

2. Main results

The main result of this paper is the following:

Theorem 2.1. *Let g be a continuous function which is non-decreasing on $[a, b]$, $0 \le a \le b < \infty$, with $g(x) > 0$ for $x > 0$. Suppose that $p \ge q \ge 1$, $0 < q + s \le p$, $\delta > 0$ and $f(x)$ is non-negative, non-decreasing and Lebesgue-Stieltjes integrable with respect to $g(x)$ on $[a, b]$. Then*

$$\begin{aligned}(2.1)\quad &\left[\int_a^b\left[\int_x^b f(t)dg(t)\right]^q f(x)^s v_1(x)dg(x) + B(p,q,s,\delta,g(a))\right]^{\frac{1}{q+s}} \\ &\le C(p,q,s,\delta)^{\frac{1}{q+s}}\left[\int_a^b f(t)^p u_1(t)dg(t)\right]^{\frac{1}{p}},\end{aligned}$$

where

$$(2.2)\qquad \begin{aligned} v_1(x) =& g(x)^{\frac{\delta(q+s)}{p}-1}\left[g(x)^{-\delta}-g(b)^{-\delta}\right]^{\frac{(q+s)-pq}{p}}, \\ u_1(t) =& g(t)^{\frac{pq(1+\delta)}{q+s}-1}, \\ C(p,q,s,\delta) =& \left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}}\left[\frac{p}{q+s}\right] \text{ and} \\ B(p,q,s,\delta,g(a)) =& C(p,q,s,\delta)g(a)^{\frac{\delta(q+s)}{p}}\left[\int_a^b g(t)^{\frac{pq}{(q+s)}(1+\delta)-(1+\delta)}f(t)^p dg(t)\right]^{\frac{q+s}{p}}. \end{aligned}$$

Proof. Through the proof of the theorem, we will use the following adaptations of Jensen's inequality for convex functions.
For $p \geq q \geq 1$,

$$(2.3)\qquad \left[\int_x^b h(x,t)^{\frac{1}{pq}} d\lambda(t)\right]^q \leq \left[\int_x^b d\lambda(t)\right]^{q-1}\left[\int_x^b h(x,t)^{\frac{1}{p}} d\lambda(t)\right]$$

and for $0 < \tau \leq 1$

$$(2.4)\qquad \int_x^b h(x,t)d\lambda(t) \leq \left[\int_x^b g(t)^{-(1+\delta)}dg(t)\right]^{1-\tau}\left[\int_x^b h(x,t)^{\frac{1}{\tau}} d\lambda(t)\right]^{\tau}$$

where $h(x,t) \geq 0$ for $x \geq 0$, $t \geq 0$ and λ is non-decreasing in $[a,b]$ (see [1] and [3]).
Letting

$$h(x,t) = g(x)^{\delta(q+s)}g(t)^{pq(1+\delta)}f(t)^{pq}f(x)^{sp} \text{ and } d\lambda(t) = g(t)^{-(1+\delta)}dg(t) \text{ in (2.3),}$$

we get

$$(2.5)\qquad \begin{aligned} &g(x)^{\frac{\delta(q+s)}{p}}f(x)^s\left[\int_x^b f(t)dg(t)\right]^q \\ &\leq \left[\delta^{-1}\right]^{q-1}\left[g(x)^{-\delta}-g(b)^{-\delta}\right]^{q-1}g(x)^{\frac{\delta(q+s)}{p}}f(x)^s\left[\int_x^b g(t)^{(1+\delta)(q-1)}f(t)^q dg(t)\right]. \end{aligned}$$

Since $s \geq 0$, we have $f(x)^s \leq f(t)^s$, $\forall t \in [x,b]$ and so we can write (2.5) as

$$(2.6)\qquad \begin{aligned} &g(x)^{\frac{\delta(q+s)}{p}}f(x)^s\left[\int_x^b f(t)dg(t)\right]^q \\ &\leq \left[\delta^{-1}\right]^{q-1}\left[g(x)^{-\delta}-g(b)^{-\delta}\right]^{q-1}\left[\int_x^b g(x)^{\frac{\delta(q+s)}{p}}g(t)^{(1+\delta)(q-1)}f(t)^{q+s}dg(t)\right]. \end{aligned}$$

Similarly, if we put $\tau = \frac{q+s}{p}$, $h(x,t) = g(x)^{\frac{\delta(q+s)}{p}}f(t)^{q+s}g(t)^{q(1+\delta)}$ and $d\lambda(t) = g(t)^{-(1+\delta)}dg(t)$ in (2.4) above, we have from the integral on the RHS of

(2.6) that,

$$\begin{aligned}&\int_x^b g(x)^{\frac{\delta(q+s)}{p}} g(t)^{(1+\delta)(q-1)} f(t)^{q+s} dg(t)\\ &\leq \left[\delta^{-1}\right]^{1-\frac{(q+s)}{p}} \left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{1-\frac{(q+s)}{p}} g(x)^{\frac{\delta(q+s)}{p}} \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}.\end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$\begin{aligned}&g(x)^{\frac{\delta(q+s)}{p}} f(x)^s \left[\int_x^b f(t) dg(t)\right]^q\\ &\leq \left[\delta^{-1}\right]^{\frac{pq-(q+s)}{p}} \left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{pq-(q+s)}{p}}\\ &\times g(x)^{\frac{\delta(q+s)}{p}} \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}.\end{aligned} \tag{2.8}$$

Rearranging and then integrating both sides over $[a, b]$ with respect to $g(x)^{-1}dg(x)$, we obtain

$$\begin{aligned}&\int_a^b g(x)^{\frac{\delta(q+s)}{p}-1} \left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{(q+s)-pq}{p}} \left[\int_x^b f(t) dg(t)\right]^q f(x)^s dg(x)\\ &\leq \left[\delta^{-1}\right]^{\frac{pq-(q+s)}{p}} \int_a^b g(x)^{\frac{\delta(q+s)}{p}-1} \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}} dg(x).\end{aligned} \tag{2.9}$$

Now take the RHS of (2.9) and then integrate by parts to obtain

$$\begin{aligned}&\left[\delta^{-1}\right]^{\frac{pq-(q+s)}{p}} \int_a^b g(x)^{\frac{\delta(q+s)}{p}-1} \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}} dg(x)\\ &=(-) \left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right] g(a)^{\frac{\delta(q+s)}{p}} \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}\\ &+\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \int_a^b g(x)^{\frac{pq(1+\delta)}{q+s}-1} f(x)^p \left[g(x)^\delta \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]\right]^{\frac{q+s}{p}-1} dg(x).\end{aligned} \tag{2.10}$$

Now use the fact that for $\delta > 0$, $g(x)^\delta \leq g(t)^\delta$, $\forall t \in [x, b]$ in 2.10 to obtain

(2.11)
$$\begin{aligned}
&\left[\delta^{-1}\right]^{\frac{pq-(q+s)}{p}} \int_a^b g(x)^{\frac{\delta(q+s)}{p}-1} \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}} dg(x)\\
\leq&(-)\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right] g(a)^{\frac{\delta(q+s)}{p}} \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}\\
+&\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \int_a^b g(x)^{\frac{pq(1+\delta)}{q+s}-1} f(x)^p \left[\left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-1} f(t)^p dg(t)\right]\right]^{\frac{q+s}{p}-1} dg(x).
\end{aligned}$$

Evaluating the last integral on the RHS of the inequality above, we have

$$\begin{aligned}
&\left[\delta^{-1}\right]^{\frac{pq-(q+s)}{p}} \int_a^b g(x)^{\frac{\delta(q+s)}{p}-1} \left[\int_x^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}} dg(x)\\
\leq&(-)\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right] g(a)^{\frac{\delta(q+s)}{p}} \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}\\
+&\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right] \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-1} f(t)^p dg(t)\right]^{\frac{q+s}{p}}.
\end{aligned} \tag{2.12}$$

Combining (2.9) and (2.12), we have

$$\begin{aligned}
&\int_a^b g(x)^{\frac{\delta(q+s)}{p}-1} \left[g(x)^{-\delta} - g(b)^{-\delta}\right]^{\frac{(q+s)-pq}{p}} \left[\int_x^b f(t) dg(t)\right]^q f(x)^s dg(x)\\
\leq&(-)\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right] g(a)^{\frac{\delta(q+s)}{p}} \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-(1+\delta)} f(t)^p dg(t)\right]^{\frac{q+s}{p}}\\
+&\left[\delta^{-1}\right]^{\frac{pq-(q+s)+p}{p}} \left[\frac{p}{q+s}\right] \left[\int_a^b g(t)^{\frac{pq(1+\delta)}{q+s}-1} f(t)^p dg(t)\right]^{\frac{q+s}{p}}.
\end{aligned} \tag{2.13}$$

Rearrange (2.13) and then use the notations (2.2) to obtain

$$\begin{aligned}
&\int_a^b \left[\int_x^b f(t) dg(t)\right]^q v_1(x) f(x)^s dg(x) + B(p, q, s, \delta, g(a))\\
\leq&C(p, q, s, \delta) \left[\int_a^b u_1(t) f(t)^p dg(t)\right]^{\frac{q+s}{p}}.
\end{aligned} \tag{2.14}$$

Taking the $(q+s)^{th}$ root of both sides of inequality (2.14), we obtain the assertion (2.1).

This completes the proof of the theorem. $\square$

3. Remark

Remark 3.1. When $l'(x)$ in (1.1) and $f(x)$ in (1.4) are non-decreasing, then these inequalities become special cases of (2.1). We see this by letting

(i) $f(x) = |l'(x)|$, $l(x) = \int_x^b f(t)dg(t)$, $v_1(x) = v(x)$, $u_1(x) = u(x)$, $g(x) = x$ and $g(a) = 0$, whence the inequality (2.1) reduces to the (1.1).

(ii) $s = 0$, whence (2.1) reduces to (1.4).

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