KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 35 NUMBER 1 (2011), PAGES 151–163.

MULTIPARTITE DIGRAPHS AND MARK SEQUENCES

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ABSTRACT. A k-partite 2-digraph (or briefly multipartite 2-digraph(M2D)) is an orientation of a k-partite multigraph that is without loops and contains at most 2 edges between any pair of vertices from distinct parts. Let $D = D(X_1, X_2, \ldots, X_k)$ be a k-partite 2-digraph with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, $1 \le i \le k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$, $1 \le j \le n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$ (or simply p_{ij}) $= 2\left(\sum_{t=1,t\neq i}^{k} n_t\right) + d^+_{x_{ij}} - d^-_{x_{ij}}$ as the mark (or r-score) of x_{ij}. In this paper, we characterize the marks of k-partite 2-digraphs and obtain constructive and existence criterion for k sequences of non-negative integers in non-decreasing order to be the mark sequences of some k-partite 2-digraph.

1. INTRODUCTION

A 2-digraph is an orientation of a multigraph that is without loops and contains at most 2 edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let D be a 2-digraph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and let $d^+_{v_i}$ and $d^-_{v_i}$ denote the outdegree and indegree, respectively, of a vertex v_i. Define p_{v_i} (or simply p_i) $= 2(n-1) + d^+_{v_i} - d^-_{v_i}$ as the mark (or 2-score) of v_i, so that $0 \le p_{v_i} \le 4(n-1)$. Then the sequence $P = [p_i]_1^n$ in non-decreasing order is called the mark sequence of D. Various results on marks in digraphs can be found in [6] and [7] and stronger inequalities for marks in digraphs can be seen in [4] and [8]. The results on scores in oriented graphs can be found in [1], [2] and [3]. The following result by Pirzada and Samee [5] characterizes marks in 2-digraphs.

Key words and phrases. Multipartite digraphs, Oriented graphs, Tournaments, Mark sequences, Oriented triples.
2010 *Mathematics Subject Classification.* Primary: 05C20.
Received: July 20, 2010.

Theorem 1.1. *A non-decreasing sequence $P = [p_i]_1^n$ of non-negative integers is the mark sequence of a 2-digraph if and only if for $1 \le t \le n$*

$$\sum_{i=1}^{t} p_i \ge 2t(t-1),$$

with equality when $t = n$.

A k-partite 2-digraph (or briefly multipartite 2-digraph (M2D)) is an orientation of a k-partite multigraph that is without loops and contains at most 2 edges between any pair of vertices from distinct parts. So k-partite 1-digraph is an oriented k-partite graph, and a complete k-partite 1-digraph is a k-partite tournament. Let $D = D(X_1, X_2, \ldots, X_k)$ be an M2D with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, $1 \le i \le k$. Let $d^+_{x_{ij}}$ and $d^-_{x_{ij}}$, $1 \le j \le n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$ (or simply p_{ij}) $= 2\left(\sum_{t=1, t\neq i}^{k} n_t\right) + d^+_{x_{ij}} - d^-_{x_{ij}}$ as the mark (or 2-score) of x_{ij}. Clearly, $0 \le p_{x_{ij}} \le 4\sum_{t=1,t\neq i}^{k} n_t$. Then the k sequences $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, in non-decreasing order are called the mark sequences of D.

An M2D can be interpreted as a result of a competition among k teams in which each player of one team plays with every player of the other $k-1$ teams at most 2 times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player x_{ij} receives a total of $p_{x_{ij}}$ points. The k sequences of non-negative integers p_i, $1 \le i \le k$, in non-decreasing order are said to be realizable if there exists an M2D with mark sequences P_i.

For two vertices x_{ij} in X_i and x_{st} in X_s, $i \neq s$ in an M2D $D(X_1, X_2, \ldots, X_k)$, we have one of the following six possibilities. (i) exactly two arcs directed from x_{ij} to x_{st} and no arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(2-0)x_{st}$, (ii) exactly two arcs directed from x_{st} to x_{ij} and no arc directed from x_{ij} to x_{st}, this is denoted by $x_{ij}(0-2)x_{st}$, (iii) exactly one arc directed from x_{ij} to x_{st} and exactly one arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(1-1)x_{st}$, and is called a pair of symmetric arcs between x_{ij} and x_{st}, (iv) exactly one arc directed from x_{ij} to x_{st} and no arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(1-0)x_{st}$, (v) exactly one arc directed from x_{st} to x_{ij} and no arc directed from x_{ij} to x_{st}, this is denoted by $x_{ij}(0-1)x_{st}$, (vi) no arc directed from x_{ij} to x_{st} and no arc directed from x_{st} to x_{ij}, this is denoted by $x_{ij}(0-0)x_{st}$.

A triple in M2D (k-partite 2-digraph) ($k \ge 3$) is an induced 2-subdigraph of three vertices with exactly one vertex from one part, and is of the form

$x_{ij}(a_1-a_2)x_{mn}(b_1-b_2)x_{st}(c_1-c_2)x_{ij}$, $(i \neq m \neq s,\ 1 \leq j \leq n_i,\ 1 \leq n \leq n_m,$ $1 \leq t \leq n_s)$, where for $1 \leq g \leq 2$, $0 \leq a_g \leq r$, $0 \leq b_g \leq 2$, $0 \leq c_g \leq 2$ and $0 \leq \sum_{g=1}^{2} a_g \leq 2$, $0 \leq \sum_{g=1}^{2} b_g \leq 2$, $0 \leq \sum_{g=1}^{2} c_g \leq 2$.

An oriented triple in M2D is an induced 1-subdigraph of three vertices with exactly one vertex from one part. An oriented triple is said to be transitive if it is of the form $x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-1)x_{ij}$, or $x_{ij}(1-0)x_{mn}(0-1)x_{st}(0-0)x_{ij}$, or $x_{ij}(1-0)x_{mn}(0-0)x_{st}(0-1)x_{ij}$, or $x_{ij}(1-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$, or $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$, otherwise it is intransitive. An M2D is said to be transitive if every of its oriented triple is transitive. In particular, a triple C in M2D is transitive if every oriented triple of C is transitive.

2. Mark sequences in multipartite digraphs

Throughout this paper we discuss k-partite 2-digraphs, with $k \geq 3$, except at few places where we require bipartite 2-digraphs. In fact we start with some observations about bipartite 2-digraphs, as these will be required in the application of algorithm obtained from Theorem 2.10. We know if $P = [p_1, p_2, \ldots, p_l]$ and $Q = [q_1, q_2, \ldots, q_m]$ are mark sequences of a bipartite 2-digraph, then $p_i \leq 4m$, $1 \leq i \leq l$ and $q_j \leq 4l$, $1 \leq j \leq m$. Also the sequences of non-negative integers $[p_1]$ and $[q_1, q_2, \ldots, q_m]$, with $p_1 + q_1 + q_2 + \ldots + q_m = 4n$ are always mark sequences of some bipartite 2-digraph. Obviously the sequences $[0]$ and $[4, 4, \ldots, 4]$ are the mark sequences of a bipartite 2-digraph.

Lemma 2.1. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 0]$ with each $p_i = 4m$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0]$ are also mark sequences of some bipartite 2-digraph.*

Lemma 2.2. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, q_m]$ with $4m - p_l = 3$, $q_m \geq 3$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 3]$ are also mark sequences of some bipartite 2-digraph.*

Proof. Let P and Q as given above be mark sequences of bipartite 2-digraph D with parts $X = \{x_1, x_2, \ldots, x_{l-1}, x_l\}$ and $Y = \{y_1, y_2, \ldots, y_{m-1}, x_m\}$. Since $4m - p_l = 3$ and $3 \leq q_m \leq 4l$, therefore in D necessarily $x_l(2-0)y_i$, for all $1 \leq i \leq m-1$. Also $y_m(1-0)x_l$, because if $y_m(0-0)x_l$, or $y_m(0-2)x_l$,or $y_m(0-1)x_l$, then in all these cases $p_{x_l} \geq 4(m-1)+2$, a contradiction to our assumption. Also $y_m(2-0)x_l$ is not possible because in that case $p_{x_l} = 4(m-1) < 4m-3$.

Now delete x_l, obviously this keeps marks of y_1,y_2, …,y_{m-1} as zeros and reduces mark of y_m by 3, and we obtain a bipartite 2-digraph with mark sequences $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 3]$, as required. □

Lemma 2.3. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, q_m]$ with $4m - p_l = 4$, $q_m \geq 4$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 4]$ are also mark sequences of some bipartite 2-digraph.*

Proof. Let P and Q as given above be mark sequences of bipartite 2-digraph D with parts $X = \{x_1, x_2, \ldots, x_{l-1}, x_l\}$ and $Y = \{y_1, y_2, \ldots, y_{m-1}, x_m\}$. Since $4m - p_l = 4$ and $4 \leq q_m \leq 4l$, therefore in D necessarily $x_l(2-0)y_i$, for all $1 \leq i \leq m-1$. Also $y_m(2-0)x_l$, because if $y_m(0-0)x_l$, or $y_m(1-0)x_l$, or $y_m(0-2)x_l$,or $y_m(0-1)x_l$, then in all these cases $p_{x_l} \geq 4(m-1)+1$, a contradiction to our assumption.

Now delete x_l, obviously this keeps marks of y_1,y_2, …,y_{m-1} as zeros and reduces mark of y_m by 4, and we obtain a bipartite 2-digraph with mark sequences $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 4]$, as required. □

Lemma 2.4. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, q_m]$ with $4m - p_l = 4$, $q_m \geq 3$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 3]$ are also mark sequences of some bipartite 2-digraph.*

Lemma 2.5. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 1, 3]$ with $4m - p_l = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.*

Lemma 2.6. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 1, 1, 2]$ with $4m - p_l = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.*

Lemma 2.7. *If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 1, 1, 1, 1]$ with $4m - p_l = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.*

Now we have the following observation about k-partite 2-digraphs, $k \geq 3$.

Lemma 2.8. *Let D and D' be two M2D's with the same mark sequences. Then D can be transformed to D' by successively transforming (i) appropriate oriented triples formed by vertices $x_{ij} \in X_i$, $x_{mn} \in X_m$ and $x_{st} \in X_s$, $i \neq m \neq s$, in one of the*

following ways:
either (a) by changing an intransitive oriented triple $x_{ij}(1-0)x_{mn}(1-0)x_{st}(1-0)x_{ij}$ *to a transitive oriented triple* $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}$*, which has same mark sequences, or vice versa,*
or (b) by changing an intransitive oriented triple $x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-0)x_{ij}$ *to a transitive oriented triple* $x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-1)x_{ij}$*, which has same mark sequences, or vice versa,*
or (ii) by changing the symmetric arcs $x_{ij}(1-1)x_{mn}$ *to* $x_{ij}(0-0)x_{mn}$*, which has same mark sequences, or vice versa.*

Proof. Let P_i be mark sequences of an M2D D whose parts are X_i, $1 \leq i \leq k$. Suppose D' be an M2D with parts X'_i, $1 \leq i \leq k$. To prove the result it is sufficient to show that D' can be obtained from D by transforming oriented triples in any one of the ways as given in i(a) or i(b) or by changing the arcs as given in (ii).

We fix n_i for $2 \leq i \leq k$ and use induction on n_1. For $n_1 = 1$, $n_2 = 1$, …, $n_k = 1$ and $k = 3$ the result is obvious. Assume that the result is true when there are fewer than n_1 vertices in the first part. Let $j_2, j_3, \ldots, j_k$ be such that for $m_2, m_3, \ldots, m_k$, $1 \leq j_i < m_i \leq n_i$ $(2 \leq i \leq k)$, the corresponding arcs have same orientations in D and D'. For $j_2, j_3, \ldots, j_k$, $2 \leq i, p, q \leq k$, $p \neq q$, the oriented triples are of the form
(I) $x_{1n_1}(1-0)x_{ij_p}(1-0)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_p}(0-0)x'_{ij_q}$
(II) $x_{1n_1}(0-0)x_{ij_p}(0-1)x_{ij_q}$ and $x'_{1n_1}(1-0)x'_{ij_p}(0-0)x'_{ij_q}$
(III) $x_{1n_1}(1-0)x_{ij_p}(0-0)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_p}(0-1)x'_{ij_q}$
(IV)$x_{1n_1}(1-0)x_{ij_p}$ and $x'_{1n_1}(0-0)x'_{ij_p}$
Case (I). Since x_{1n_1} and x'_{1n_1} have equal marks, therefore $x_{1n_1}(0-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$, or $x_{1n_1}(0-0)x_{ij_q}$ and $x'_{1n_1}(1-0)x'_{ij_q}$. Thus there is an oriented triple $x_{1n_1}(1-0)x_{ij_p}(1-0)x_{ij_q}(1-0)x_{1n_1}$, or $x_{1n_1}(1-0)x_{ij_p}(1-0)x_{ij_q}(0-0)x_{1n_1}$ in D and corresponding to these $x'_{1n_1}(0-0)x'_{ij_p}(0-0)x'_{ij_q}(0-0)x'_{1n_1}$, or $x'_{1n_1}(0-0)x'_{ij_p}(0-0)x'_{ij_q}(0-1)x'_{1n_1}$ respectively is an oriented triple in D'.
Case II. Since x_{1n_1} and x'_{1n_1} have equal marks, so $x_{1n_1}(1-0)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$ and thus there is an oriented triple $x_{1n_1}(0-0)x_{ij_p}(0-1)x_{ij_q}(0-1)x_{1n_1}$ in D and corresponding to this $x'_{1n_1}(1-0)x'_{ij_p}(0-0)x'_{ij_q}(0-0)x'_{1n_1}$ is an oriented triple in D'.
Case III. Since x_{1n_1} and x'_{1n_1} have equal marks, so $x_{1n_1}(0-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$ and thus there is an oriented triple $x_{1n_1}(1-0)x_{ij_p}(0-0)x_{ij_q}(1-0)x_{1n_1}$ in D and corresponding to this $x'_{1n_1}(0-0)x'_{ij_p}(0-1)x'_{ij_q}(0-0)x'_{1n_1}$ is an oriented triple in D'.

Case IV. Since x_{1n_1} and x'_{1n_1} have equal marks, so $x_{1n_1}(1-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$.

Thus it follows from (I)-(IV) that there is an M2D that can be obtained from D by any one of the transformations i(a) or i(b) or (ii) with mark sequences remaining unchanged. Hence the result follows by induction. □

Lemma 2.8 leads to the following observation.

Corollary 2.1. *Among all M2D's with given mark sequences those with the fewest arcs are transitive.*

A transmitter is a vertex with indegree zero. We assume without loss of generality that transitive M2D's have no arcs of the form $x(1-1)y$, as they can be transformed to $x(0-0)y$ with same marks. This implies that in a transitive M2D with mark sequences $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, any of the vertex with mark p_{in_i} can act as transmitter.

Let $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, be k sequences of non-negative integers in non-decreasing order with $p_{1n_1} \ge p_{in_i}$,

$$2\sum_{t=2}^{k} n_t \le p_{1n_1} \le 4\sum_{t=2}^{k} n_t \quad \text{and} \quad 0 \le p_{in_i} \le 4\left(\sum_{t=2, t\ne i}^{k} n_t\right) - 3$$

for all $2 \le i \le k$. Let P'_1 be obtained from P_1 by deleting one entry p_{1n_1}, and let $P'_2, P'_3, \ldots, P'_k$ be obtained as follows.

(A)(i). If $p_{1n_1} \ge 3\sum_{t=2}^{k} n_t$, then reducing $4\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ largest entries of $P_2, P_3, \cdots, P_k$ by one each,

or **(ii).** If $p_{1n_1} < 3\sum_{t=2}^{k} n_t$, then reducing $3\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ largest entries of $P_2, P_3, \ldots, P_k$ by two each, and $p_{1n_1} - 2\left(\sum_{t=2}^{k} n_t\right)$ remaining entries by one each.

(B). In case any one of $p_{in_i} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$, $2 \le i \le k$, say for instance $p_{jn_j} = 4\sum_{t=2}^{k} n_t - 2$, then also $p_{1n_1} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$ as $p_{1n_1} \ge p_{in_i}$. In this case we reduce p_{jn_j} by two.

The next result provides a useful recursive test whether the sequences of non-negative integers form the mark sequences of some M2D.

Theorem 2.1. *P_i are the mark sequences of some M2D if and only if P'_i (arranged in non-decreasing order) as obtained in (A) or (B) are the mark sequences of some M2D.*

Proof. Let P'_i, $1 \le i \le k$, be the mark sequences of some M2D $D'(X'_1, X'_2, \ldots, X'_k)$. First assume $P'_2, P'_3, \ldots, P'_k$ be obtained from $P_2, P_3, \ldots, P_k$ as in (A)(i). Construct an M2D $D(X_1, X_2, \ldots, X_k)$ as follows. Let $X_1 = X_1' \cup \{x\}, X_i = X_i'$, $2 \leq i \leq k$, with $X_1' \cap \{x\} = \phi$. Let $x(1-0)y$ for those vertices y of $X_2^{/}, X_3', \ldots X_k'$ whose marks are reduced by one in going from P_i to P_i', and $x(2-0)y$ for those vertices y of $X_2', X_3', \ldots, X_k'$ whose marks are not reduced in going from P_i to P_i', $1 \leq i \leq k$. Then $D(X_1, X_2, \ldots, X_k)$ is M2D with mark sequences P_i, $1 \leq i \leq k$.

Now, if P_2', P_3', …, P_k' are obtained from P_2, P_3,… , P_k as in (A)(ii), then construct an M2D $D(X_1, X_2, \ldots, X_k)$ as follows. Let $X_1 = X_1' \cup \{x\}$, $X_i = X_i'$, $2 \leq i \leq k$, with $X_1' \cap \{x\} = \phi$. Let $x(1-0)y$ for those vertices y of $X_2', X_3', \ldots, X_k'$ whose marks are reduced by one in going from P_i to P_i', and $x(1-1)y$ for those vertices y of $X_2', X_3', \ldots, X_k'$ whose marks are reduced by two in going from P_i to P_i', $1 \leq i \leq k$. For (B), we take $x(1-1)y$ for those vertices y of $X_2', X_3', \ldots, X_k'$ whose marks are reduced by two in going from P_i to P_i', $1 \leq i \leq k$. Then $D(X_1, X_2, \ldots, X_k)$ is M2D with mark sequences P_i, $1 \leq i \leq k$.

Conversely, suppose P_i be mark sequences of some M2D $D(X_1, X_2, \ldots, X_k)$, $1 \leq i \leq k$. Now any of the vertex $x_{in_i} \in X_i$ with mark p_{in_i}, $1 \leq i \leq k$, can act as a transmitter. Clearly for (i) and (ii) $p_{1n_1} \geq 2\sum_{t=2}^{k} n_t$ and $p_{in_i} \leq 4\sum_{t=1,t\neq i}^{k} n_t - 3$ for all $2 \leq i \leq k$, because if $p_{1n_1} \leq 2\sum_{t=2}^{k} n_t$, then by deleting p_{1n_1} we have to reduce more than $\sum_{t=2}^{k} n_t$ entries from $P_2, P_3, \ldots, P_k$, which is absurd.
(i) If $p_{1n_1} \geq 3\sum_{t=2}^{k} n_t$, let X be the set of $4\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ vertices of largest marks in $X_2, X_3, \ldots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. In case X does not contain all $4\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ vertices of largest marks, we can bring them to X by using Lemma 2.8. Construct $D(X_1, X_2, \ldots, X_k)$ such that $x_{1n_1}(1-0)x$ for all x in X and $x_{1n_1}(2-0)y$ for all y in Y. Clearly, $D(X_1, X_2, \ldots, X_k) - \{x_{1n_1}\}$ realizes $P_1', P_2', \ldots, P_k'$.
(ii) If $p_{1n_1} < 3\sum_{t=2}^{k} n_t$, let X be the set of $3\left(\sum_{t=2}^{k} n_t\right) - p_{1n_1}$ vertices of largest marks in $X_2, X_3, \ldots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. Construct $D(X_1, X_2, \ldots, X_k)$ such that $x_{1n_1}(1-1)x$ for all x in X and $x_{1n_1}(1-0)y$ for all y in Y. Then again $D(X_1, X_2, \ldots, X_k) - \{x_{1n_1}\}$ realizes $P_1', P_2', \ldots, P_k'$.
(B) If for instance $p_{jn_j} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$, then necessarily $p_{1n_1} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$ so that $x_{1n_1}(0-0)x_{jn_j}$ or $x_{1n_1}(1-1)x_{jn_j}$. Clearly, $D(X_1, X_2, \ldots, X_k) - \{x_{1n_1}\}$ realizes $P_1', P_2', \ldots, P_k'$. □

Theorem 2.1 provides an algorithm for determining whether or not the k sequences P_i, $1 \leq i \leq k$, of non-negative integers in non-decreasing order are mark sequences, and for constructing a corresponding M2D. Let $P_i = [p_{i1}, p_{i2}, \ldots, p_{in_i}]$, $1 \leq i \leq k$, with (a) $p_{1n_1} \geq 2\sum_{t=2}^{k} n_t$, (b) $p_{in_i} \leq 4\left(\sum_{t=1,t\neq i}^{k} n_t\right) - 2$ for all $2 \leq i \leq k$, be mark sequences of an M2D with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, $1 \leq i \leq k$. Deleting p_{1n_1} and performing A(i) or A(ii), or B of Theorem 2.1 according as $p_{1n_1} \geq 3\sum_{t=2}^{k} n_t$ or $p_{1n_1} < 3\sum_{t=2}^{k} n_t$, or any one of $p_{in_i} = 4\left(\sum_{t=2}^{k} n_t\right) - 2$, $2 \leq i \leq k$, we obtain $P_2', P_3', \ldots, P_k'$. If the marks of the vertices x_{ij} were decreased by one in this process, then the construction yielded $x_{1n_1}(1-0)x_{ij}$, and if these were decreased by two, then the construction yielded $x_{1n_1}(1-1)x_{ij}$. For vertices x_{st} whose marks remained unchanged, the construction yielded $x_{1n_1}(2-0)x_{st}$. Note that if any of the conditions A or B does not hold, then we delete p_{in_i} for that i for which the conditions get satisfied, and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not P_i are mark sequences, and if P_i are mark sequences, then an M2D with mark sequences P_i, $1 \leq i \leq k$ is constructed. During the application of Theorem 2.1, the algorithm may reach a stage where we get just two sequences, and it is not possible to apply Theorem 2.1, in those cases we apply Lemma 2.1 to Lemma 2.7.

We illustrate this reduction and the resulting construction with the following examples.

Example 2.1. Consider the five sequences of non-negative integers as follows:
$P_1 = [15, 16, 21]$, $P_2 = [16, 20]$, $P_3 = [15, 20]$, $P_4 = [17, 19]$, $P_5 = [16, 17]$.

1. [15,16], [15,18], [14,18], [16,17], [15,16]
$x_{13}(0-0)x_{22}$, $x_{13}(0-0)x_{32}$, $x_{13}(0-0)x_{42}$, $x_{13}(1-0)x_{21}$, $x_{13}(1-0)x_{31}$, $x_{13}(1-0)x_{41}$, $x_{13}(1-0)x_{51}$, $x_{13}(1-0)x_{52}$
2. [15], [13,16], [12,16], [14,15], [13,14]
$x_{12}(0-0)x_{21}$, $x_{12}(0-0)x_{22}$, $x_{12}(0-0)x_{31}$, $x_{12}(0-0)x_{32}$, $x_{12}(0-0)x_{41}$, $x_{12}(0-0)x_{42}$, $x_{12}(0-0)x_{51}$, $x_{12}(1-0)x_{52}$
3. [13], [13], [11,14], [12,13], [12,12]
$x_{22}(0-0)x_{32}$, $x_{22}(0-0)x_{11}$, $x_{22}(0-0)x_{42}$, $x_{22}(0-0)x_{41}$, $x_{22}(0-0)x_{52}$, $x_{22}(1-0)x_{31}$, $x_{22}(1-0)x_{51}$
4. [11], [11], [11], [10,11], [11,11]
$x_{32}(0-0)x_{11}$, $x_{32}(0-0)x_{21}$, $x_{32}(0-0)x_{42}$, $x_{32}(0-0)x_{41}$, $x_{32}(1-0)x_{51}$, $x_{32}(1-0)x_{52}$
5. [9], [9], [9], [10], [9,10]
$x_{42}(0-0)x_{11}$, $x_{42}(0-0)x_{21}$, $x_{42}(0-0)x_{31}$, $x_{42}(0-0)x_{52}$, $x_{42}(1-0)x_{51}$

6. [7], [8], [8], [8], [9]
$x_{51}(0-0)x_{41}$, $x_{51}(0-0)x_{11}$, $x_{51}(1-0)x_{21}$, $x_{51}(1-0)x_{31}$
7. [5], ϕ, [6], [6], [7], $x_{21}(0-0)x_{52}$, $x_{21}(0-0)x_{31}$, $x_{21}(0-0)x_{41}$, $x_{21}(0-0)x_{11}$
8. [3], ϕ, ϕ, [4], [5], $x_{31}(0-0)x_{52}$, $x_{31}(0-0)x_{41}$, $x_{31}(0-0)x_{11}$
9. [1], ϕ, ϕ, ϕ, [3], $x_{41}(0-0)x_{52}$, $x_{41}(0-0)x_{11}$
10. [0], ϕ, ϕ, ϕ, ϕ, $x_{52}(0-0)x_{11}$

The resulting 5-partite 2-digraph has mark sequences $P_1 = [15,16,21]$, $P_2 = [16,20]$, $P_3 = [15,20]$, $P_4 = [17,19]$, $P_5 = [16,17]$ with vertex sets $X_1 = \{x_{11}, x_{12}, x_{13}\}$, $X_2 = \{x_{21}, x_{22}\}$, $X_3 = \{x_{31}, x_{32}\}$, $X_4 = \{x_{41}, x_{42}\}$, $X_5 = \{x_{51}, x_{52}\}$, and arcs as $x_{13}(0-0)x_{22}$, $x_{13}(0-0)x_{32}$, $x_{13}(0-0)x_{42}$, $x_{13}(1-0)x_{21}$, $x_{13}(1-0)x_{31}$, $x_{13}(1-0)x_{41}$, $x_{13}(1-0)x_{51}$, $x_{13}(1-0)x_{52}$, $x_{12}(0-0)x_{21}$, $x_{12}(0-0)x_{22}$, $x_{12}(0-0)x_{31}$, $x_{12}(0-0)x_{32}$, $x_{12}(0-0)x_{41}$, $x_{12}(0-0)x_{42}$, $x_{12}(0-0)x_{51}$, $x_{12}(1-0)x_{52}$, $x_{22}(0-0)x_{32}$, $x_{22}(0-0)x_{11}$, $x_{22}(0-0)x_{42}$, $x_{22}(0-0)x_{41}$, $x_{22}(0-0)x_{52}$, $x_{22}(1-0)x_{31}$, $x_{22}(1-0)x_{51}$, $x_{32}(0-0)x_{11}$, $x_{32}(0-0)x_{21}$, $x_{32}(0-0)x_{42}$, $x_{32}(0-0)x_{41}$, $x_{32}(1-0)x_{51}$, $x_{32}(1-0)x_{52}$, $x_{42}(0-0)x_{11}$, $x_{42}(0-0)x_{21}$, $x_{42}(0-0)x_{31}$, $x_{42}(0-0)x_{52}$, $x_{42}(1-0)x_{51}$, $x_{51}(0-0)x_{41}$, $x_{51}(0-0)x_{11}$, $x_{51}(1-0)x_{21}$, $x_{51}(1-0)x_{31}$, $x_{21}(0-0)x_{52}$, $x_{21}(0-0)x_{31}$, $x_{21}(0-0)x_{41}$, $x_{21}(0-0)x_{11}$, $x_{31}(0-0)x_{52}$, $x_{31}(0-0)x_{41}$, $x_{31}(0-0)x_{11}$, $x_{41}(0-0)x_{52}$, $x_{41}(0-0)x_{11}$, $x_{52}(0-0)x_{11}$.

Example 2.2. Consider the three sequences of non-negative integers as follows:
$P_1 = [12,18]$, $P_2 = [1,2,3]$, $P_3 = [10,18]$.
1. [12], [1,2,3], [10,16]
$x_{12}(2-0)x_{21}$, $x_{12}(2-0)x_{22}$, $x_{12}(2-0)x_{23}$, $x_{12}(2-0)x_{31}$, $x_{12}(0-0)x_{32}$
2. [12], [1,2,3], [10]
$x_{32}(2-0)x_{11}$, $x_{32}(2-0)x_{21}$, $x_{32}(2-0)x_{22}$, $x_{32}(2-0)x_{23}$
3. ϕ, [1,2,1], [8]
$x_{11}(2-0)x_{21}$, $x_{11}(2-0)x_{22}$, $x_{11}(0-0)x_{23}$, $x_{11}(0-0)x_{31}$
3. ϕ, [0,0,0], ϕ
$x_{31}(1-0)x_{21}$, $x_{31}(0-0)x_{22}$, $x_{31}(1-0)x_{23}$

The resulting 3-partite 2-digraph has mark sequences $P_1 = [12,18]$, $P_2 = [1,2,3]$, $P_3 = [10,18]$ and vertex sets $X_1 = \{x_{11}, x_{12}\}$, $X_2 = \{x_{21}, x_{22}, x_{23}\}$, $X_3 = \{x_{31}, x_{32}\}$ and arcs $x_{12}(2-0)x_{21}$, $x_{12}(2-0)x_{22}$, $x_{12}(2-0)x_{23}$, $x_{12}(2-0)x_{31}$, $x_{12}(0-0)x_{32}$, $x_{32}(2-0)x_{11}$, $x_{32}(2-0)x_{21}$, $x_{32}(2-0)x_{22}$, $x_{32}(2-0)x_{23}$, $x_{11}(2-0)x_{21}$, $x_{11}(2-0)x_{22}$, $x_{11}(0-0)x_{23}$, $x_{11}(0-0)x_{31}$, $x_{31}(1-0)x_{21}$, $x_{31}(0-0)x_{22}$, $x_{31}(1-0)x_{23}$.

The next result gives a combinatorial criterion for determining whether k sequences of non-negative integers in non-decreasing order are realizable as marks.

Theorem 2.2. *Let $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, be k sequences of non-negative integers in non-decreasing order. Then, P_i are the mark sequences of some M2D if and only if*

$$\sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} \ge 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j, \tag{2.1}$$

for all sequences of k integers s_i, $1 \le s_i \le n_i$, with equality when $s_i = n_i$ for all i.

Proof. A sub k-partite 2-digraph induced by s_i vertices for $1 \le i \le k$, $1 \le s_i \le n_i$, has a sum of marks $4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j$. This proves the necessity.

For sufficiency, let $P_i = [p_{ij}]_1^{n_i}$, $1 \le i \le k$, be the sequences of non-negative integers in non-decreasing order satisfying conditions (1) but are not the mark sequences of any M2D. Let these sequences be chosen in such a way that n_i, $1 \le i \le k$, be smallest possible and p_{11} is the least with that choice of n_i. We consider the following two cases.

Case (i). Assume equality in (1) holds for some $s_j \le n_j$, $1 \le j \le k-1$, $s_k < n_k$, so that

$$\sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} = 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_i s_j.$$

By the minimality of n_i, $1 \le i \le k$, the sequences $P_i = [P_{i1}, P_{i2}, \dots, P_{is_i}]$ are mark sequences of some M2D $D'(X_1', X_2', \dots, X_k')$.

Define

$$P_i'' = \left[\left(p_{i(s_i+1)} - 4\sum_{t=1,t\ne i}^{k} s_t\right), \left(p_{i(s_i+2)} - 4\sum_{t=1,t\ne i}^{k} s_t\right), \dots, \left(p_{i(n_i)} - 4\sum_{t=1,t\ne i}^{k} s_t\right)\right],$$

$1 \le i \le k$.

Now consider the sum

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{f_i}[p_{i(s_i+j)} - 4\sum_{t=1,t\ne i}^{k} s_t] \\
&\qquad = \sum_{i=1}^{k}\sum_{j=1}^{f_i} p_{i(s_i+j)} - 4\sum_{i=1}^{k}\sum_{j=1}^{f_i}\sum_{t=1,t\ne i}^{k} s_t \\
&\qquad = \sum_{i=1}^{k}\sum_{j=1}^{f_i+s_i} p_{ij} - \sum_{i=1}^{k}\sum_{j=1}^{s_i} p_{ij} - 4\sum_{i=1}^{k}\sum_{j=1}^{f_i}\sum_{i=1}^{k} s_t + 4\sum_{i=1}^{k}\sum_{j=1}^{f_i} s_i
\end{aligned}$$

$$\geq 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}[(s_i+f_i)(s_j+f_j)] - 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_is_j$$

$$-4\sum_{i=1}^{k} f_i\sum_{t=1}^{k} s_t + 4\sum_{i=1}^{k} f_is_i$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(s_is_j + s_if_j + f_is_j + f_if_j) - 2r\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_is_j$$

$$-4\sum_{i=1}^{k}\sum_{t=1}^{k} f_is_t + 4\sum_{i=1}^{k} f_is_i$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_is_j + 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(s_if_j + f_is_j) + 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_if_j$$

$$-4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} s_is_j - 4\sum_{i=1}^{k}\sum_{t=1}^{k} f_is_t + 4\sum_{i=1}^{k} f_is_i$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_if_j$$

$$+4\sum_{i=1}^{k-1}[(s_if_{i+1}) + f_is_{i+1}) + (s_if_{i+2}) + f_is_{i+2}) + \ldots + (s_if_k) + f_is_k)]$$

$$-4\sum_{i=1}^{k}(f_is_1 + f_is_2 + \ldots + f_is_k) + 4(f_1s_1 + f_2s_2 + \ldots + f_ks_k)$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_if_j$$

$$+4\{[(s_1f_2 + f_1s_2) + (s_1f_3 + f_1s_3) + \ldots + (s_1f_k + f_1s_k)]$$

$$+[(s_2f_3 + f_2s_3) + (s_2f_4 + f_2s_4) + \ldots + (s_2f_k + f_2s_k)]$$

$$+\ldots + [(s_{k-1}f_k + f_{k-1}s_k)]\}$$

$$-4[(f_1s_1 + f_1s_2 + \ldots + f_1s_k) + (f_2s_1 + f_2s_2 + \ldots + f_2s_k)$$

$$+\ldots + (f_ks_1 + f_ks_2 + \ldots + f_ks_k)]$$

$$+4(f_1s_1 + f_2s_2 + \ldots + f_ks_k)$$

$$= 4\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} f_if_j,$$

for $1 \leq f_i \leq n_i - s_i$ with equality when $f_i = n_i - s_i$ for all i, $1 \leq i \leq k$. Then by minimality of n_i, $1 \leq i \leq k$, the sequences P_i'' form the mark sequences of some M2D $D''(X_1'', X_2'', \ldots, X_k'')$.

Now construct a new M2D $D(X_1, X_2, \ldots, X_k)$ as follows. Let

$$X_1 = X_1' \cup X_1'', X_2 = X_2' \cup X_2'', \ldots, X_k = X_k' \cup X_k''$$

with $X_i' \cap X_i'' = \phi$.

Let

$$x_i''(2-0)x_1', x_i''(2-0)x_2', \ldots, x_i''(2-0)x_{i-1}', x_i''(2-0)x_{i+1}', \ldots, x_i''(2-0)x_k',$$

for all x_i'' in X_i'' and for all x_i' in X_i', $1 \leq i \leq k$. Then clearly $D(X_1, X_2, \ldots, X_k)$ is an M2D with mark sequences P_i, $1 \leq i \leq k$, which is a contradiction.

Case (ii). Assume strict inequality in (1) holds for some $s_i \neq n_i$, $1 \leq i \leq k$. Let

$$P_1' = [p_{11} - 1, p_{12}, \ldots, p_{1n_1-1}, p_{1n_1} + 1]$$

and

$$P_j' = [p_{j1}, p_{j2}, \ldots, p_{jn_j}]$$

for all j, $2 \leq j \leq k$. Clearly the sequences P_i', $1 \leq i \leq k$, satisfy conditions (1). Therefore by the minimality of p_{11}, the sequences P_i', $1 \leq i \leq k$, are mark sequences of some M2D $D'(X_1', X_2', \ldots, X_k')$. Let

$$p_{x_{11}} = p_{11} - 1$$

and

$$p_{x_{1n_1}} = p_{1n_1} + 1.$$

Since

$$p_{x_{1n_1}} > p_{x_{11}} + 1,$$

there exists a vertex x_{ij} in X_i, $2 \leq i \leq k$, $1 \leq j \leq n_i$, such that $x_{1n_1}(1-0)x_{ij}(1-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(1-0)x_{11}$, or $x_{1n_1}(1-0)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(0-0)x_{11}$ in $D'(X_1', X_2', \ldots, X_k')$, and if these are changed to $x_{1n_1}(0-0)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-1)x_{ij}(0-0)x_{11}$, or $x_{1n_1}(0-0)x_{ij}(0-1)x_{11}$, or $x_{1n_1}(0-1)x_{ij}(0-1)x_{11}$ respectively, the result is an M2D with mark sequences P_i, $1 \leq i \leq k$, which is again a contradiction. Hence the result follows. □

References

[1] P. Avery, *Score sequences of oriented graphs*, J. Graph Theory **15 (3)** (1991), 251–257.

[2] S. Pirzada and Merajuddin, *Score lists of oriented tripartite graphs*, Novi Sad J. Math. **26 (2)** (1996), 1-9.

[3] S. Pirzada, Merajuddin and Yin Jianhu, *On the score lists of oriented bipartite graphs*, J. Math. Study **33 (4)** (2000), 354–359.

[4] S. Pirzada and T. A. Naikoo, *Inequalities for marks in digraphs*, J. Math. Inequalities and Appl. **9 (2)** (2006), 189–198.
[5] S. Pirzada and U. Samee, *Mark sequences in digraphs*, Seminare Loth. de Combinatorie **55** (2006), ArtB55c.
[6] S. Pirzada, *Mark sequences in multidigraphs*, Discrete Math. and Appl. **17 (1)** (2007), 71–76.
[7] S. Pirzada and T. A. Naikoo, *Mark sets in 2-digraphs*, J. Applied and Comp. Math, **10 (1)** (2011), To appear.
[8] S. Pirzada, U. Samee, Merajuddin, and T. A. Naikoo, *Inequalities for mark in multidigraphs*, Italian J. Pure and Appl. Math. To appear.

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