MULTIPARTITE DIGRAPHS AND MARK SEQUENCES

UMATUL SAMEE

ABSTRACT. A k-partite 2-digraph (or briefly multipartite 2-digraph(M2D)) is an orientation of a k-partite multigraph that is without loops and contains at most 2 edges between any pair of vertices from distinct parts. Let \( D = D(X_1, X_2, \ldots, X_k) \) be a k-partite 2-digraph with parts \( X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}, \) \( 1 \leq i \leq k. \) Let \( d^+_x, d^-_x, \) and \( d^+_{x_{ij}}, d^-_{x_{ij}} \) be respectively the outdegree and indegree of a vertex \( x_{ij} \in X_i. \) Define \( p_{x_{ij}} \) (or simply \( p_{ij} \)) as the mark (or \( r \)-score) of \( x_{ij}. \) In this paper, we characterize the marks of \( k \)-partite 2-digraphs and obtain constructive and existence criterion for \( k \) sequences of non-negative integers in non-decreasing order to be the mark sequences of some \( k \)-partite 2-digraph.

1. Introduction

A 2-digraph is an orientation of a multigraph that is without loops and contains at most 2 edges between any pair of distinct vertices. So, 1-digraph is an oriented graph, and a complete 1-digraph is a tournament. Let \( D \) be a 2-digraph with vertex set \( V = \{v_1, v_2, \ldots, v_n\}, \) and let \( d^+_v, d^-_v \) denote the outdegree and indegree, respectively, of a vertex \( v_i. \) Define \( p_v \) (or simply \( p_i \)) as the mark (or 2-score) of \( v_i, \) so that \( 0 \leq p_v \leq 4(n-1). \) Then the sequence \( P = [p_i]_1^n \) in non-decreasing order is called the mark sequence of \( D. \) Various results on marks in digraphs can be found in [6] and [7] and stronger inequalities for marks in digraphs can be seen in [4] and [8]. The results on scores in oriented graphs can be found in [1], [2] and [3]. The following result by Pirzada and Samee [5] characterizes marks in 2-digraphs.

Key words and phrases. Multipartite digraphs, Oriented graphs, Tournaments, Mark sequences, Oriented triples.

2010 Mathematics Subject Classification. Primary: 05C20.

Received: July 20, 2010.

151
Theorem 1.1. A non-decreasing sequence $P = [p_i]_1^n$ of non-negative integers is the mark sequence of a 2-digraph if and only if for $1 \leq t \leq n$

$$\sum_{i=1}^{t} p_i \geq 2t(t - 1),$$

with equality when $t = n$.

A $k$-partite 2-digraph (or briefly multipartite 2-digraph (M2D)) is an orientation of a $k$-partite multigraph that is without loops and contains at most 2 edges between any pair of vertices from distinct parts. So $k$-partite 1-digraph is an oriented $k$-partite graph, and a complete $k$-partite 1-digraph is a $k$-partite tournament. Let $D = D(X_1, X_2, \ldots, X_k)$ be an M2D with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, $1 \leq i \leq k$. Let $d^+_x$ and $d^-_x$, $1 \leq j \leq n_i$, be respectively the outdegree and indegree of a vertex $x_{ij} \in X_i$. Define $p_{x_{ij}}$ (or simply $p_{ij}$) $= 2 \sum_{t=1,t\neq_i}^{k} n_t + d^+_x - d^-_x$ as the mark (or 2-score) of $x_{ij}$. Clearly, $0 \leq p_{x_{ij}} \leq 4 \sum_{t=1,t\neq_i}^{k} n_t$. Then the $k$ sequences $P_i = [p_{ij}]_1^n$, $1 \leq i \leq k$, in non-decreasing order are called the mark sequences of $D$.

An M2D can be interpreted as a result of a competition among $k$ teams in which each player of one team plays with every player of the other $k - 1$ teams at most 2 times in which ties (draws) are allowed. A player receives two points for each win, and one point for each tie. With this marking system, player $x_{ij}$ receives a total of $p_{x_{ij}}$ points. The $k$ sequences of non-negative integers $p_i$, $1 \leq i \leq k$, in non-decreasing order are said to be realizable if there exists an M2D with mark sequences $P_i$.

For two vertices $x_{ij}$ in $X_i$ and $x_{st}$ in $X_s$, $i \neq s$ in an M2D $D(X_1, X_2, \ldots, X_k)$, we have one of the following six possibilities. (i) exactly two arcs directed from $x_{ij}$ to $x_{st}$ and no arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(2 - 0)x_{st}$, (ii) exactly two arcs directed from $x_{st}$ to $x_{ij}$ and no arc directed from $x_{ij}$ to $x_{st}$, this is denoted by $x_{ij}(0 - 2)x_{st}$, (iii) exactly one arc directed from $x_{ij}$ to $x_{st}$ and exactly one arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(1 - 1)x_{st}$, and is called a pair of symmetric arcs between $x_{ij}$ and $x_{st}$, (iv) exactly one arc directed from $x_{ij}$ to $x_{st}$ and no arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(1 - 0)x_{st}$, (v) exactly one arc directed from $x_{st}$ to $x_{ij}$ and no arc directed from $x_{ij}$ to $x_{st}$, this is denoted by $x_{ij}(0 - 1)x_{st}$, (vi) no arc directed from $x_{ij}$ to $x_{st}$ and no arc directed from $x_{st}$ to $x_{ij}$, this is denoted by $x_{ij}(0 - 0)x_{st}$.

A triple in M2D ($k$-partite 2-digraph) ($k \geq 3$) is an induced 2-subdigraph of three vertices with exactly one vertex from one part, and is of the form
Let \( x_{ij}(a_1 - a_2)x_{mn}(b_1 - b_2)x_{st}(c_1 - c_2)x_{ij} \) \((i \neq m \neq s, 1 \leq j \leq n_i, 1 \leq n \leq n_m, 1 \leq t \leq n_s)\), where for \( 1 \leq g \leq 2, 0 \leq a_g \leq r, 0 \leq b_g \leq 2, 0 \leq c_g \leq 2 \) and \( 0 \leq \sum_{g=1}^{2} a_g \leq 2, 0 \leq \sum_{g=1}^{2} b_g \leq 2, 0 \leq \sum_{g=1}^{2} c_g \leq 2 \).

An oriented triple in M2D is an induced 1-subdigraph of three vertices with exactly one vertex from one part. An oriented triple is said to be transitive if it is of the form \( x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-1)x_{ij}, \) or \( x_{ij}(1-0)x_{mn}(0-1)x_{st}(0-0)x_{ij}, \) or \( x_{ij}(1-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}, \) or \( x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij}, \) otherwise it is intransitive. An M2D is said to be transitive if every of its oriented triple is transitive. In particular, a triple \( C \) in M2D is transitive if every oriented triple of \( C \) is transitive.

2. MARK SEQUENCES IN MULTIPARTITE DIGRAPHS

Throughout this paper we discuss \( k \)-partite 2-digraphs, with \( k \geq 3 \), except at few places where we require bipartite 2-digraphs. In fact we start with some observations about bipartite 2-digraphs, as these will be required in the application of algorithm obtained from Theorem 2.10. We know if \( P = [p_1, p_2, \ldots, p_l] \) and \( Q = [q_1, q_2, \ldots, q_m] \) are mark sequences of a bipartite 2-digraph, then \( p_i \leq 4m, 1 \leq i \leq l \) and \( q_j \leq 4l, 1 \leq j \leq m \). Also the sequences of non-negative integers \([p_1]\) and \([q_1, q_2, \ldots, q_m]\), with \( p_1 + q_1 + q_2 + \ldots + q_m = 4n \) are always mark sequences of some bipartite 2-digraph. Obviously the sequences \([0]\) and \([4, 4, \ldots, 4]\) are the mark sequences of a bipartite 2-digraph.

**Lemma 2.1.** If \( P = [p_1, p_2, \ldots, p_{l-1}, p_l] \) and \( Q = [0, 0, \ldots, 0, 0] \) with each \( p_i = 4m \) are mark sequences of some bipartite 2-digraph, then \( P' = [p_1, p_2, \ldots, p_{l-1}] \) and \( Q' = [0, 0, \ldots, 0] \) are also mark sequences of some bipartite 2-digraph.

**Lemma 2.2.** If \( P = [p_1, p_2, \ldots, p_{l-1}, p_l] \) and \( Q = [0, 0, \ldots, 0, q_m] \) with \( 4m - p_1 = 3, q_m \geq 3 \) are mark sequences of some bipartite 2-digraph, then \( P' = [p_1, p_2, \ldots, p_{l-1}] \) and \( Q' = [0, 0, \ldots, 0, q_m - 3] \) are also mark sequences of some bipartite 2-digraph.

**Proof.** Let \( P \) and \( Q \) as given above be mark sequences of bipartite 2-digraph \( D \) with parts \( X = \{x_1, x_2, \ldots, x_{l-1}, x_l\} \) and \( Y = \{y_1, y_2, \ldots, y_{m-1}, y_m\} \). Since \( 4m - p_1 = 3 \) and \( 3 \leq q_m \leq 4l \), therefore in \( D \) necessarily \( x_i(2-0)y_i \), for all \( 1 \leq i \leq m - 1 \). Also \( y_m(1-0)x_l \), because if \( y_m(0-0)x_l \), or \( y_m(0-2)x_l \), or \( y_m(0-1)x_l \), then in all these cases \( p_{x_i} \geq 4(m-1) + 2 \), a contradiction to our assumption. Also \( y_m(2-0)x_l \) is not possible because in that case \( p_{x_i} = 4(m-1) < 4m - 3 \).
Now delete $x_l$, obviously this keeps marks of $y_1, y_2, \ldots, y_{m-1}$ as zeros and reduces mark of $y_m$ by 3, and we obtain a bipartite 2-digraph with mark sequences $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 3]$, as required. \hfill \Box

**Lemma 2.3.** If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, q_m]$ with $4m - p_l = 4$, $q_m \geq 4$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 4]$ are also mark sequences of some bipartite 2-digraph.

**Proof.** Let $P$ and $Q$ as given above be mark sequences of bipartite 2-digraph $D$ with parts $X = \{x_1, x_2, \ldots, x_{l-1}, x_l\}$ and $Y = \{y_1, y_2, \ldots, y_{m-1}, y_m\}$. Since $4m - p_l = 4$ and $4 \leq q_m \leq 4l$, therefore in $D$ necessarily $x_l(2 - 0)y_i$, for all $1 \leq i \leq m - 1$. Also $y_m(2 - 0)x_l$ because if $y_m(0 - 0)x_l$, or $y_m(1 - 0)x_l$, or $y_m(0 - 2)x_l$, or $y_m(0 - 1)x_l$, then in all these cases $p_{x_l} \geq 4(m - 1) + 1$, a contradiction to our assumption.

Now delete $x_l$, obviously this keeps marks of $y_1, y_2, \ldots, y_{m-1}$ as zeros and reduces mark of $y_m$ by 4, and we obtain a bipartite 2-digraph with mark sequences $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 4]$, as required. \hfill \Box

**Lemma 2.4.** If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, q_m]$ with $4m - p_l = 4$, $q_m \geq 3$ are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, q_m - 3]$ are also mark sequences of some bipartite 2-digraph.

**Lemma 2.5.** If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 1, 3]$ with $4m - p_l = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

**Lemma 2.6.** If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 1, 1, 2]$ with $4m - p_l = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

**Lemma 2.7.** If $P = [p_1, p_2, \ldots, p_{l-1}, p_l]$ and $Q = [0, 0, \ldots, 0, 1, 1, 1, 1]$ with $4m - p_l = 4$, are mark sequences of some bipartite 2-digraph, then $P' = [p_1, p_2, \ldots, p_{l-1}]$ and $Q' = [0, 0, \ldots, 0, 0, 0]$ are also mark sequences of some bipartite 2-digraph.

Now we have the following observation about $k$-partite 2-digraphs, $k \geq 3$.

**Lemma 2.8.** Let $D$ and $D'$ be two M2D's with the same mark sequences. Then $D$ can be transformed to $D'$ by successively transforming (i) appropriate oriented triples formed by vertices $x_{ij} \in X_i$, $x_{mn} \in X_m$ and $x_{st} \in X_s$, $i \neq m \neq s$, in one of the
following ways:
either (a) by changing an intransitive oriented triple \( x_{ij}(1-0)x_{mn}(1-0)x_{st}(1-0)x_{ij} \)
to a transitive oriented triple \( x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-0)x_{ij} \), which has same mark
sequences, or vice versa,
or (b) by changing an intransitive oriented triple \( x_{ij}(1-0)x_{mn}(1-0)x_{st}(0-0)x_{ij} \)
to a transitive oriented triple \( x_{ij}(0-0)x_{mn}(0-0)x_{st}(0-1)x_{ij} \), which has same mark
sequences, or vice versa,

or (ii) by changing the symmetric arcs \( x_{ij}(1-1)x_{mn} \) to \( x_{ij}(0-0)x_{mn} \), which has same mark
sequences, or vice versa.

Proof. Let \( P_i \) be mark sequences of an M2D \( D \) whose parts are \( X_i \), \( 1 \leq i \leq k \).
Suppose \( D' \) be an M2D with parts \( X'_i \), \( 1 \leq i \leq k \). To prove the result it is sufficient
to show that \( D' \) can be obtained from \( D \) by transforming oriented triples in any one
of the ways as given in i(a) or i(b) or by changing the arcs as given in (ii).

We fix \( n_i \) for \( 2 \leq i \leq k \) and use induction on \( n_1 \). For \( n_1 = 1, n_2 = 1, \ldots, n_k = 1 \)
and \( k = 3 \) the result is obvious. Assume that the result is true when there are fewer
than \( n_1 \) vertices in the first part. Let \( j_2, j_3, \ldots, j_k \) be such that for \( m_2, m_3, \ldots, m_k, \)
\( 1 \leq j_i < m_i \leq n_i \) (\( 2 \leq i \leq k \)), the corresponding arcs have same orientations in \( D \)
and \( D' \). For \( j_2, j_3, \ldots, j_k, 2 \leq i, p, q \leq k, p \neq q \), the oriented triples are of the form
(I) \( x_{1m_1}(1-0)x_{ij'}(1-0)x_{ijq} \) and \( x'_{1m_1}(0-0)x'_{ij'}(0-0)x'_{ijq} \)
(II) \( x_{1m_2}(0-0)x_{ij'}(0-1)x_{ijq} \) and \( x'_{1m_2}(1-0)x'_{ij'}(0-0)x'_{ijq} \)
(III) \( x_{1m_3}(1-0)x_{ij'}(0-0)x_{ijq} \) and \( x'_{1m_3}(0-0)x'_{ij'}(0-1)x'_{ijq} \)
(IV) \( x_{1m_4}(1-0)x_{ij'} \) and \( x'_{1m_4}(0-0)x'_{ij'} \)

Case (I). Since \( x_{1m_1} \) and \( x'_{1m_1} \) have equal marks, therefore \( x_{1m_1}(0-1)x_{ijq} \) and \( x'_{1m_1}(0-0)x'_{ijq} \), or \( x_{1m_1}(0-0)x_{ijq} \) and \( x'_{1m_1}(1-0)x'_{ijq} \). Thus there is an oriented triple \( x_{1m_1}(1-0)x_{ij}x_{1m_1}(1-0)x_{ijq} \) in \( D \) and corresponding
to these \( x'_{1m_1}(0-0)x'_{ij}x'_{1m_1}(0-0)x'_{ijq} \), or \( x'_{1m_1}(0-0)x'_{ij}x'_{1m_1}(0-1)x'_{ijq} \) respectively is an oriented triple in \( D' \).

Case II. Since \( x_{1m_1} \) and \( x'_{1m_1} \) have equal marks, so \( x_{1m_1}(1-0)x_{ijq} \) and \( x'_{1m_1}(0-0)x'_{ijq} \)
and thus there is an oriented triple \( x_{1m_1}(0-0)x_{ijq}(0-1)x_{ijq}(0-1)x_{1m_1} \) in \( D \) and corresponding
to this \( x'_{1m_1}(1-0)x'_{ijq}(0-0)x'_{ijq}(0-0)x'_{1m_1} \) is an oriented triple in \( D' \).

Case III. Since \( x_{1m_1} \) and \( x'_{1m_1} \) have equal marks, so \( x_{1m_1}(0-1)x_{ijq} \) and \( x'_{1m_1}(0-0)x'_{ijq} \)
and thus there is an oriented triple \( x_{1m_1}(1-0)x_{ij}(0-0)x_{ij}(0-1)x_{1m_1} \) in \( D \) and corresponding
to this \( x'_{1m_1}(0-0)x'_{ij}(0-1)x'_{ij}(0-0)x'_{1m_1} \) is an oriented triple in \( D' \).
Case IV. Since $x_{1n_1}$ and $x'_{1n_1}$ have equal marks, so $x_{1n_1}(1-1)x_{ij_q}$ and $x'_{1n_1}(0-0)x'_{ij_q}$.

Thus it follows from (I)-(IV) that there is an M2D that can be obtained from $D$ by any one of the transformations i(a) or i(b) or (ii) with mark sequences remaining unchanged. Hence the result follows by induction. □

Lemma 2.8 leads to the following observation.

Corollary 2.1. Among all M2D’s with given mark sequences those with the fewest arcs are transitive.

A transmitter is a vertex with indegree zero. We assume without loss of generality that transitive M2D’s have no arcs of the form $x(1-1)y$, as they can be transformed to $x(0-0)y$ with same marks. This implies that in a transitive M2D with mark sequences $P_i = [p_{ij}]_{n_i}$, $1 \leq i \leq k$, any of the vertex with mark $p_{in_i}$ can act as transmitter.

Let $P_i = [p_{ij}]_{n_i}$, $1 \leq i \leq k$, be $k$ sequences of non-negative integers in non-decreasing order with $p_{1n_1} \geq p_{in_i}$, $2 \leq i \leq k$. Let $P'_i$ be obtained from $P_i$ by deleting one entry $p_{1n_1}$, and let $P'_2, P'_3, \ldots, P'_k$ be obtained as follows.

(A)(i). If $p_{1n_1} \geq 3 \sum_{t=2}^{k} n_t$, then reducing $4 \left( \sum_{t=2}^{k} n_t \right) - p_{1n_1}$ largest entries of $P_2, P_3, \ldots, P_k$ by one each,
or (ii). If $p_{1n_1} < 3 \sum_{t=2}^{k} n_t$, then reducing $3 \left( \sum_{t=2}^{k} n_t \right) - p_{1n_1}$ largest entries of $P_2, P_3, \ldots, P_k$ by two each, and $p_{1n_1} - 2 \left( \sum_{t=2}^{k} n_t \right)$ remaining entries by one each.

(B). In case any one of $p_{in_i} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$, $2 \leq i \leq k$, say for instance $p_{jn_j} = 4 \sum_{t=2}^{k} n_t - 2$, then also $p_{1n_1} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$ as $p_{1n_1} \geq p_{in_i}$. In this case we reduce $p_{jn_j}$ by two.

The next result provides a useful recursive test whether the sequences of non-negative integers form the mark sequences of some M2D.

Theorem 2.1. $P_i$ are the mark sequences of some M2D if and only if $P'_i$ (arranged in non-decreasing order) as obtained in (A) or (B) are the mark sequences of some M2D.

Proof. Let $P'_i$, $1 \leq i \leq k$, be the mark sequences of some M2D $D'(X'_1, X'_2, \ldots, X'_k)$. First assume $P'_2, P'_3, \ldots, P'_k$ be obtained from $P_2, P_3, \ldots, P_k$ as in (A)(i). Construct
an M2D $D(X_1, X_2, \ldots, X_k)$ as follows. Let $X_i = X_i' \cup \{x\}, X_i = X_i', 2 \leq i \leq k$, with $X_i' \cap \{x\} = \emptyset$. Let $x(1 - 0)y$ for those vertices $y$ of $X'_2, X'_3, \ldots, X'_k$ whose marks are reduced by one in going from $P_i$ to $P'_i$, and $x(2 - 0)y$ for those vertices $y$ of $X'_2, X'_3, \ldots, X'_k$ whose marks are not reduced in going from $P_i$ to $P'_i$, $1 \leq i \leq k$. Then $D(X_1, X_2, \ldots, X_k)$ is M2D with mark sequences $P_i, 1 \leq i \leq k$.

Now, if $P'_2, P'_3, \ldots, P'_k$ are obtained from $P_2, P_3, \ldots, P_k$ as in (A)(ii), then construct a M2D $D(X_1, X_2, \ldots, X_k)$ as follows. Let $X_i = X_i' \cup \{x\}, X_i = X_i', 2 \leq i \leq k$, with $X_i' \cap \{x\} = \emptyset$. Let $x(1 - 0)y$ for those vertices $y$ of $X'_2, X'_3, \ldots, X'_k$ whose marks are reduced by one in going from $P_i$ to $P'_i$, and $x(1 - 1)y$ for those vertices $y$ of $X'_2, X'_3, \ldots, X'_k$ whose marks are reduced by two in going from $P_i$ to $P'_i$, $1 \leq i \leq k$. For (B), we take $x(1 - 1)y$ for those vertices $y$ of $X'_2, X'_3, \ldots, X'_k$ whose marks are reduced by two in going from $P_i$ to $P'_i$, $1 \leq i \leq k$. Then $D(X_1, X_2, \ldots, X_k)$ is M2D with mark sequences $P_i, 1 \leq i \leq k$.

Conversely, suppose $P_i$ be mark sequences of some M2D $D(X_1, X_2, \ldots, X_k), 1 \leq i \leq k$. Any of the vertex $x_{im} \in X_i$ with mark $p_{im}, 1 \leq i \leq k$, can act as a transmitter. Clearly for (i) and (ii) $p_{1m} \geq 2 \sum_{t=2}^{k} n_t$ and $p_{1m} \leq 4 \sum_{t=1, t \neq i}^{k} n_t - 3$ for all $2 \leq i \leq k$, because if $p_{1m} \leq 2 \sum_{t=2}^{k} n_t$, then by deleting $p_{1m}$ we have to reduce more than $\sum_{t=2}^{k} n_t$ entries from $P_2, P_3, \ldots, P_k$, which is absurd.

(i) If $p_{1m} \geq 3 \sum_{t=2}^{k} n_t$, let $X$ be the set of $4 \left( \sum_{t=2}^{k} n_t \right) - p_{1m}$ vertices of largest marks in $X_2, X_3, \ldots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. In case $X$ does not contain all $4 \left( \sum_{t=2}^{k} n_t \right) - p_{1m}$ vertices of largest marks, we can bring them to $X$ by using Lemma 2.8. Construct $D(X_1, X_2, \ldots, X_k)$ such that $x_{1m}(1 - 0)x$ for all $x$ in $X$ and $x_{1m}(2 - 0)y$ for all $y$ in $Y$. Clearly, $D(X_1, X_2, \ldots, X_k) - \{x_{1m}\}$ realizes $P'_1, P'_2, \ldots, P'_k$.

(ii) If $p_{1m} \leq 3 \sum_{t=2}^{k} n_t$, let $X$ be the set of $3 \left( \sum_{t=2}^{k} n_t \right) - p_{1m}$ vertices of largest marks in $X_2, X_3, \ldots, X_k$ and let $Y = \cup_{t=2}^{k} X_t - X$. Construct $D(X_1, X_2, \ldots, X_k)$ such that $x_{1m}(1 - 1)x$ for all $x$ in $X$ and $x_{1m}(1 - 0)y$ for all $y$ in $Y$. Then again $D(X_1, X_2, \ldots, X_k) - \{x_{1m}\}$ realizes $P'_1, P'_2, \ldots, P'_k$.

(B) If for instance $x_{jn} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$, then necessarily $p_{1m} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2$ so that $x_{1m}(0 - 0)x_{jn}$ or $x_{1m}(1 - 1)x_{jn}$. Clearly, $D(X_1, X_2, \ldots, X_k) - \{x_{1m}\}$ realizes $P'_1, P'_2, \ldots, P'_k$. \qed

Theorem 2.1 provides an algorithm for determining whether or not the $k$ sequences $P_i, 1 \leq i \leq k$, of non-negative integers in non-decreasing order are mark sequences, and for constructing a corresponding M2D. Let $P_i = [p_{i1}, p_{i2}, \ldots, p_{im}], 1 \leq i \leq k,$
with (a) \( p_{1n_1} \geq 2 \sum_{t=2}^{k} n_t \), (b) \( p_{m_i} \leq 4 \left( \sum_{t=1, t \neq i}^{k} n_t \right) - 2 \) for all \( 2 \leq i \leq k \), be mark sequences of an M2D with parts \( X_i = \{ x_{i1}, x_{i2}, \ldots, x_{im_i} \} \), \( 1 \leq i \leq k \). Deleting \( p_{1n_1} \) and performing A(i) or A(ii), or B of Theorem 2.1 according as \( p_{1n_1} \geq 3 \sum_{t=2}^{k} n_t \) or \( p_{1n_1} < 3 \sum_{t=2}^{k} n_t \), or any one of \( p_{m_i} = 4 \left( \sum_{t=2}^{k} n_t \right) - 2 \), \( 2 \leq i \leq k \), we obtain \( P'_2, P'_3, \ldots, P'_k \). If the marks of the vertices \( x_{ij} \) were decreased by one in this process, then the construction yielded \( x_{1n_1}(1 - 0)x_{ij} \), and if these were decreased by two, then the construction yielded \( x_{1n_1}(1 - 1)x_{ij} \). For vertices \( x_{st} \) whose marks remained unchanged, the construction yielded \( x_{1n_1}(2 - 0)x_{st} \). Note that if any of the conditions A or B does not hold, then we delete \( p_{m_i} \) for that \( i \) for which the conditions get satisfied, and the same argument is used for defining arcs. If this procedure is applied recursively, then it tests whether or not \( P_i \) are mark sequences, and if \( P_i \) are mark sequences, then an M2D with mark sequences \( P_i, 1 \leq i \leq k \) is constructed. During the application of Theorem 2.1, the algorithm may reach a stage where we get just two sequences, and it is not possible to apply Theorem 2.1, in those cases we apply Lemma 2.1 to Lemma 2.7.

We illustrate this reduction and the resulting construction with the following examples.

**Example 2.1.** Consider the five sequences of non-negative integers as follows:

\[ P_1 = [15, 16, 21], \quad P_2 = [16, 20], \quad P_3 = [15, 20], \quad P_4 = [17, 19], \quad P_5 = [16, 17]. \]

1. \([15, 16], [15, 18], [14, 18], [16, 17], [15, 16]\)
   \[ x_{13}(0-0)x_{22}, \quad x_{13}(0-0)x_{32}, \quad x_{13}(0-0)x_{42}, \quad x_{13}(1-0)x_{21}, \quad x_{13}(1-0)x_{31}, \quad x_{13}(1-0)x_{41}, \quad x_{13}(1-0)x_{51}, \quad x_{13}(1-0)x_{52} \]

2. \([15], [13, 16], [12, 16], [14, 15], [13, 14]\)
   \[ x_{12}(0-0)x_{21}, \quad x_{12}(0-0)x_{22}, \quad x_{12}(0-0)x_{32}, \quad x_{12}(0-0)x_{31}, \quad x_{12}(0-0)x_{41}, \quad x_{12}(0-0)x_{42}, \quad x_{12}(0-0)x_{51}, \quad x_{12}(1-0)x_{52} \]

3. \([13], [13], [11, 14], [12, 13], [12, 12]\)
   \[ x_{22}(0-0)x_{32}, \quad x_{22}(0-0)x_{11}, \quad x_{22}(0-0)x_{42}, \quad x_{22}(0-0)x_{41}, \quad x_{22}(0-0)x_{52}, \quad x_{22}(1-0)x_{31}, \quad x_{22}(1-0)x_{51} \]

4. \([11], [11], [11], [10, 11], [11, 11]\)
   \[ x_{32}(0-0)x_{11}, \quad x_{32}(0-0)x_{21}, \quad x_{32}(0-0)x_{42}, \quad x_{32}(0-0)x_{41}, \quad x_{32}(1-0)x_{51}, \quad x_{32}(1-0)x_{52} \]

5. \([9], [9], [9], [10], [9, 10]\)
   \[ x_{42}(0-0)x_{11}, \quad x_{42}(0-0)x_{21}, \quad x_{42}(0-0)x_{31}, \quad x_{42}(0-0)x_{52}, \quad x_{42}(1-0)x_{51} \]
6. \([7, 8, [8], [8], [9]]
\[ x_{51}(0-0)x_{41}, x_{51}(0-0)x_{11}, x_{51}(1-0)x_{21}, x_{51}(1-0)x_{31} \]
7. \([5, \phi, [6], [6], [7]]
\[ x_{21}(0-0)x_{52}, x_{21}(0-0)x_{31}, x_{21}(0-0)x_{41}, x_{21}(0-0)x_{11} \]
8. \([3, \phi, [4], [5]]
\[ x_{31}(0-0)x_{52}, x_{31}(0-0)x_{41}, x_{31}(0-0)x_{11} \]
9. \([1, \phi, \phi, [3]]
\[ x_{41}(0-0)x_{52}, x_{41}(0-0)x_{11} \]
10. \([0], \phi, \phi, \phi, x_{52}(0-0)x_{11} \]

The resulting 5-partite 2-digraph has mark sequences \(P_1 = [15, 16, 21], P_2 = [16, 20], P_3 = [15, 20], P_4 = [17, 19], P_5 = [16, 17]\) with vertex sets \(X_1 = \{x_{11}, x_{12}, x_{13}\}, X_2 = \{x_{21}, x_{22}\}, X_3 = \{x_{31}, x_{32}\}, X_4 = \{x_{41}, x_{42}\}, X_5 = \{x_{51}, x_{52}\}\), and arcs as
\[
\begin{align*}
x_{13}(0-0)x_{21}, x_{13}(0-0)x_{31}, x_{13}(1-0)x_{31}, x_{13}(1-0)x_{41}, x_{13}(1-0)x_{51}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}, x_{12}(0-0)x_{21}, x_{12}(0-0)x_{31}, x_{12}(0-0)x_{41}, x_{12}(0-0)x_{51}, x_{12}(0-0)x_{52}, x_{12}(0-0)x_{11}.
\end{align*}
\]

**Example 2.2.** Consider the three sequences of non-negative integers as follows:

1. \([12], [1,2,3], [10,16]\)
\[ x_{12}(2-0)x_{21}, x_{12}(2-0)x_{22}, x_{12}(2-0)x_{23}, x_{12}(2-0)x_{31}, x_{12}(0-0)x_{32} \]
2. \([12], [1,2,3], [10]\)
\[ x_{32}(2-0)x_{11}, x_{32}(2-0)x_{21}, x_{32}(2-0)x_{22}, x_{32}(2-0)x_{23} \]
3. \([\phi], [1,2,1], [8]\)
\[ x_{11}(2-0)x_{21}, x_{11}(2-0)x_{22}, x_{11}(0-0)x_{23}, x_{11}(0-0)x_{31} \]
4. \([\phi], [0,0,0], \phi\)
\[ x_{31}(1-0)x_{21}, x_{31}(0-0)x_{22}, x_{31}(1-0)x_{23} \]

The resulting 3-partite 2-digraph has mark sequences \(P_1 = [12, 18], P_3 = [1, 2, 3], P_5 = [10, 18]\) and vertex sets \(X_1 = \{x_{11}, x_{12}\}, X_2 = \{x_{21}, x_{22}, x_{23}\}, X_3 = \{x_{31}, x_{32}\}\), and arcs \(x_{12}(2-0)x_{21}, x_{12}(2-0)x_{22}, x_{12}(2-0)x_{23}, x_{12}(2-0)x_{31}, x_{12}(0-0)x_{32}, x_{32}(2-0)x_{11}, x_{32}(2-0)x_{21}, x_{32}(2-0)x_{22}, x_{32}(2-0)x_{23}, x_{11}(2-0)x_{21}, x_{11}(2-0)x_{22}, x_{11}(0-0)x_{23}, x_{11}(0-0)x_{31}, x_{11}(1-0)x_{21}, x_{31}(0-0)x_{22}, x_{31}(1-0)x_{23}.\)
The next result gives a combinatorial criterion for determining whether \(k\) sequences of non-negative integers in non-decreasing order are realizable as marks.

**Theorem 2.2.** Let \(P_i = [p_{ij}]_{1}^{n_i}, 1 \leq i \leq k\), be \(k\) sequences of non-negative integers in non-decreasing order. Then, \(P_i\) are the mark sequences of some M2D if and only if

\[
\sum_{i=1}^{k} \sum_{j=1}^{s_i} p_{ij} \geq 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j,
\]

for all sequences of \(k\) integers \(s_i, 1 \leq s_i \leq n_i\), with equality when \(s_i = n_i\) for all \(i\).

**Proof.** A sub \(k\)-partite 2-digraph induced by \(s_i\) vertices for \(1 \leq i \leq k\), \(1 \leq s_i \leq n_i\), has a sum of marks \(4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j\). This proves the necessity.

For sufficiency, let \(P_i = [p_{ij}]_{1}^{n_i}, 1 \leq i \leq k\), be the sequences of non-negative integers in non-decreasing order satisfying conditions (1) but are not the mark sequences of any M2D. Let these sequences be chosen in such a way that \(n_i, 1 \leq i \leq k\), be smallest possible and \(p_{i1}\) is the least with that choice of \(n_i\). We consider the following two cases.

**Case (i).** Assume equality in (1) holds for some \(s_j \leq n_j, 1 \leq j \leq k-1, s_k < n_k\), so that

\[
\sum_{i=1}^{k} \sum_{j=1}^{s_i} p_{ij} = 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j.
\]

By the minimality of \(n_i, 1 \leq i \leq k\), the sequences \(P_i = [P_{i1}, P_{i2}, \ldots, P_{is_i}]\) are mark sequences of some M2D \(D'(X'_1, X'_2, \ldots, X'_k)\).

Define

\[
P''_i = \left[ \left( p_{i(s_i+1)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right), \left( p_{i(s_i+2)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right), \ldots, \left( p_{i(n_i)} - 4 \sum_{t=1, t \neq i}^{k} s_t \right) \right],
\]

\(1 \leq i \leq k\).

Now consider the sum

\[
\sum_{i=1}^{k} \sum_{j=1}^{f_i} [p_{i(s_i+j)} - 4 \sum_{t=1, t \neq i}^{k} s_t]
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{f_i} p_{i(s_i+j)} - 4 \sum_{i=1}^{k} \sum_{j=1}^{f_i} \sum_{t=1, t \neq i}^{k} s_t
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{f_i+s_i} p_{ij} - \sum_{i=1}^{k} \sum_{j=1}^{s_i} p_{ij} - 4 \sum_{i=1}^{k} \sum_{j=1}^{f_i} \sum_{t=1}^{k} s_t + 4 \sum_{i=1}^{k} \sum_{j=1}^{f_i} s_i
\]
\[
\begin{align*}
g &\geq 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} [(s_i + f_i)(s_j + f_j)] - 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j \\
&\quad - 4 \sum_{i=1}^{k} f_i \sum_{t=1}^{k} s_t + 4 \sum_{i=1}^{k} f_is_i \\
&\quad = 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (s_is_j + s_if_j + f_is_j + f_if_j) - 2r \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j \\
&\quad - 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} s_is_j - 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_is_t + 4 \sum_{i=1}^{k} f_is_i \\
&\quad = 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j \\
&\quad + 4 \sum_{i=1}^{k} [(s_if_{i+1} + f_is_{i+1}) + (s_if_{i+2} + f_is_{i+2}) + \ldots + (s_if_k + f_is_k)] \\
&\quad - 4 \sum_{i=1}^{k} (f_is_1 + f_is_2 + \ldots + f_is_k) + 4(f_is_1 + f_is_2 + \ldots + f_is_k) \\
&\quad = 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j \\
&\quad + 4\left\{ (s_if_2 + f_is_2) + (s_if_3 + f_is_3) + \ldots + (s_if_k + f_is_k) \right\} \\
&\quad + \left\{ (s_2f_3 + f_2s_3) + (s_2f_4 + f_2s_4) + \ldots + (s_2f_k + f_2s_k) \right\} \\
&\quad + \ldots + \left\{ (s_{k-1}f_k + f_{k-1}s_k) \right\} \\
&\quad - 4\left\{ (f_is_1 + f_is_2 + \ldots + f_is_k) + (f_2s_1 + f_2s_2 + \ldots + f_2s_k) \\
&\quad + \ldots + (f_ks_1 + f_ks_2 + \ldots + f_ks_k) \right\} \\
&\quad + 4(f_is_1 + f_is_2 + \ldots + f_is_k) \\
&\quad = 4 \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} f_if_j,
\end{align*}
\]

for \(1 \leq f_i \leq n_i - s_i\) with equality when \(f_i = n_i - s_i\) for all \(i, 1 \leq i \leq k\). Then by minimality of \(n_i, 1 \leq i \leq k, the sequences P''_n form the mark sequences of some M2D D''(X''_1, X''_2, \ldots, X''_k)\).
Now construct a new M2D $D(X_1, X_2, \ldots, X_k)$ as follows. Let
$$X_1 = X'_1 \cup X''_1, X_2 = X'_2 \cup X''_2, \ldots, X_k = X'_k \cup X''_k$$
with $X'_i \cap X''_i = \phi$.

Let
$$x''_i (2 - 0)x'_1, x''_i (2 - 0)x'_2, \ldots, x''_i (2 - 0)x'_{i-1}, x''_i (2 - 0)x'_{i+1}, \ldots, x''_i (2 - 0)x'_k,$$
for all $x''_i$ in $X''_i$ and for all $x'_i$ in $X'_i, 1 \leq i \leq k$. Then clearly $D(X_1, X_2, \ldots, X_k)$ is an M2D with mark sequences $P_i, 1 \leq i \leq k$, which is a contradiction.

**Case (ii).** Assume strict inequality in (1) holds for some $s_i \neq n_i, 1 \leq i \leq k$. Let
$$P'_i = [p_{i1} - 1, p_{i2}, \ldots, p_{in1} - 1, p_{in1} + 1]$$
and
$$P'_j = [p{j1}, p{j2}, \ldots, p{jn_j}]$$
for all $j, 2 \leq j \leq k$. Clearly the sequences $P'_i, 1 \leq i \leq k$, satisfy conditions (1).

Therefore by the minimality of $p_{i1}$, the sequences $P'_i, 1 \leq i \leq k$, are mark sequences of some M2D $D'(X'_1, X'_2, \ldots, X'_k)$. Let
$$p_{x_{i1}} = p_{i1} - 1$$
and
$$p_{x_{i1n}} = p_{i1n} + 1.$$

Since
$$p_{x_{i1n}} > p_{x_{i1}} + 1,$$
there exists a vertex $x_{ij}$ in $X_i, 2 \leq i \leq k, 1 \leq j \leq n_i$, such that $x_{i1n} (1-0)x_{ij} (1-0)x_{11},$ or $x_{i1n} (0-0)x_{ij} (1-0)x_{11},$ or $x_{i1n} (1-0)x_{ij} (0-0)x_{11},$ or $x_{i1n} (0-0)x_{ij} (0-0)x_{11}$ in $D'(X'_1, X'_2, \ldots, X'_k)$, and if these are changed to $x_{i1n} (0-0)x_{ij} (0-0)x_{11},$ or $x_{i1n} (0-0)x_{ij} (0-0)x_{11},$ or $x_{i1n} (0-0)x_{ij} (0-1)x_{11},$ or $x_{i1n} (0-1)x_{ij} (0-1)x_{11}$ respectively, the result is an M2D with mark sequences $P_i, 1 \leq i \leq k$, which is again a contradiction. Hence the result follows.

**References**


Department of Mathematics,
University of Kashmir,
Srinagar, India
*E-mail address: pzsamee@yahoo.co.in*