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THE NUMBER OF SPANNING TREES OF DOUBLE GRAPHS

LIU WU-XIN [1] AND WEI FU-YI [2]

Abstract. In this article, we study the number of spanning trees of double graphs (direct product of a simple graph) and obtain some upper bounds for double graphs, especially for double tree graphs, double unicyclic graphs and double bicyclic graphs. The extremal graphs are also determined.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. For $u, v \in V(G)$, u adj v means that u and v are adjacent. Let d_i be the degree of v_i, $\Delta = \max_{1\le i\le n} d_i$, and $\Delta = \min_{1\le i\le n} d_i$. The number of spanning trees of G is denoted by $t(G)$. Recall that a connected graph with n vertices and $n-1+c$ edges is called $c-cyclic$. The $0-cyclic$ graphs are known as trees, and the $1-cyclic$ and $2-cyclic$ graphs are unicyclic graphs and bicyclic graphs, respectively. Let P_n, C_n and K_n be the path, the cycle and the complete graph with n vertices, respectively.

The *direct product* of two graphs G and H is the graph $G \times H$ with $V(G \times H) = V(G) \times V(H)$ such that (v_1, w_1) adj (v_2, w_2) in $G \times H$ if and only if v_1 adj v_2 in G and w_1 adj w_2 in H. The graph K_2^s is obtained from the complete graph K_2 by adding a loop to every vertex.

Munarini et al. [4] defined the *double* graph of a simple graph G as the graph $D[G] = G \times K_2^s$ and studied its elementary properties (see Figure 1 for some examples of double graphs). Das [1] obtained a sharp upper bound for the number of spanning

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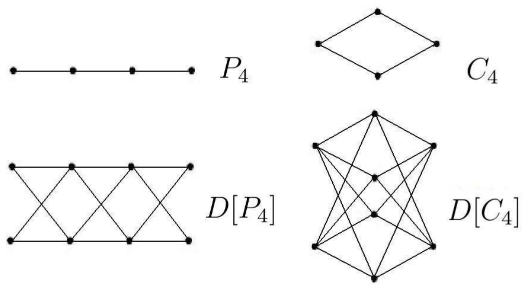

FIGURE 1. A path and its double, a cycle and its double.

trees of connected graphs. In this article we investigate the number of spanning trees of double graphs, and obtain upper bounds with an emphasis on double graphs, especially for double trees, double unicyclic graphs and double bicyclic graphs. The extremal graphs are characterized.

2. Upper bounds for the number of spanning trees of double trees and double unicyclic graphs

In this section, we give upper bounds for the number of the spanning trees of double graphs. In particular, we consider the case of double trees and the case of double unicyclic graphs. For $n = 1$, G is the graph with one isolated vertex and it is easy to see that $D[G]$ is the graph with two isolated vertices. In this section, we always assume $n \geq 2$. First, we need some lemmas:

Lemma 2.1. [4] *Let G be a simple connected graph on n vertices with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$t(D[G]) = 4^{n-1} d_1 d_2 \cdots d_n t(G).$$

Lemma 2.2. [1] *Let G be a simple connected graph with n vertices, e edges and degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$\prod_{i=1}^{n} d_i \leq \left(\frac{2e}{n}\right)^n$$

with equality if and only if $d_1 = d_2 = \ldots = d_n = \frac{2e}{n}$.

Next we give our three theorems.

Theorem 2.1. *Let G be a simple connected graph with $n \geq 3$ vertices. Then*

$$t(D[G]) \leq 4^{n-1}(n-1)^n n^{n-2}$$

with equality if and only if $G \cong K_n$.

Proof. If G is a simple graph on n vertices, then $d_i \leq n-1$ for every vertex. And since $G \subseteq K_n$, we also have $t(G) \leq t(K_n) = n^{n-2}$. By Lemmas 2.1., we have

$$t(D[G]) \leq 4^{n-1}(n-1)^n n^{n-2}$$

with equality if and only if $G \cong K_n$. □

Theorem 2.2. *Let T be a tree on $n \geq 2$ vertices with degree sequence $d_1, d_2, \ldots, d_n$. Then*

$$t(D[T]) \leq 2^{3n-4}$$

with equality if and only if $T \cong P_n$.

Proof. The tree T has at least two vertices of degrees one, say $d_{n-1} = d_n = 1$. Then $e = n-1$ and $\sum_{i=1}^{n-2} d_i = 2e-4$. By the Arithmetic-Geometric inequality, $\prod_{i=1}^{n-2} d_i \leq (\frac{2e-4}{n-2})^{n-2} = 2^{n-2}$ with equality if and only if $d_1 = d_2 = \ldots = d_{n-2} = 2$. Combining Lemma 2.1 and $t(T) = 1$, we have $t(D[T]) = 4^{n-1} d_1 d_2 \cdots d_n t(T) \leq 2^{3n-4}$ with equality if and only if $T \cong P_n$. □

Theorem 2.3. *Let U_n^l be a unicyclic graph with n vertices and cycle length l. Then*

$$t(D[U_n^l]) \leq n2^{3n-2}$$

with equality if and only if $U_n^l \cong C_n$.

Proof. For $e = n$, we have $t(U_n^l) = l$. By Lemma 2.1 and Lemma 2.2, we have

$$t(D[U_n^l]) = 4^{n-1} t(U_n^l) \prod_{i=1}^{n} d_i = 4^{n-1} l \prod_{i=1}^{n} d_i \leq 4^{n-1} l \left(\frac{2e}{n}\right)^n = l2^{3n-2} \leq n2^{3n-2}.$$

The first equality holds if and only if $d_1 = d_2 = \ldots = d_n = 2$, and the second equality holds if and only if $l = n$. Then both equalities hold if and only if $U_n^l \cong C_n$. □

3. An upper bound for the number of spanning trees of bicyclic graphs

In this section, we study the number of spanning trees of double bicyclic graphs, and determine the corresponding extremal graphs. Let B_k be a bicyclic graph with k $(4 \leq k \leq n)$ vertices.

First, we consider the bicyclic graphs B_k such that $\delta(B_k) > 1$.

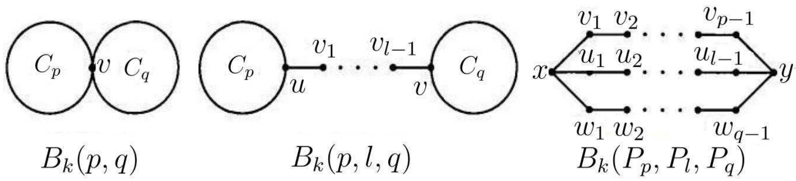

FIGURE 2. Three classes of graphs $\hat{\mathbf{B}}_{\mathbf{k}}$

For convenience, we define some graphs as in [2] and [5]. Let $\mathbf{B}_k(p,q)$ be the set of the bicyclic graphs $B_k(p,q)$ with k vertices obtained from cycles C_p and C_q by identifying a vertex C_p and a vertex of C_q, where p and q are such that $p+q-1=k$ ($k \geq 5, p \leq q$), see Figure 2. Let $\mathbf{B}_k(p,l,q)$ be the set of the bicyclic graphs $B_k(p,l,q)$ with k vertices obtained from two vertex-disjoint cycles C_p and C_q by adding the path $uv_1v_2\cdots v_{l-1}v$ with length l from the vertex u of cycle C_p to vertex v of cycle C_q, where p,l and q are such that $p+q+l-1=k$ ($k \geq 6, p \leq q, l \geq 1$), see Figure 2. Let $\mathbf{B}_k(P_p,P_l,P_q)$ be the set of the bicyclic graphs $B_k(P_p,P_l,P_q)$ with k vertices obtained from a cycle $xv_1v_2\cdots v_{p-1}yw_{q-1}\cdots w_2w_1x$ by joining vertices x and y by a path $u_1u_2\cdots u_{l-1}$ with length l, where p,l,q are such that $p+q+l-1=k$ ($k \geq 4, p \leq l \leq q$), see Figure 2.

Let $\hat{\mathbf{B}}_k = \{B_k : \delta(B_k) > 1\}$, where $4 \leq k \leq n$. It is well-known that $\hat{\mathbf{B}}_k = \mathbf{B}_k(p,q) \bigcup \mathbf{B}_k(p,l,q) \bigcup \mathbf{B}_k(P_p,P_l,P_q)$.

We have the following lemma.

Lemma 3.1. *Let $G_1, G_2, \ldots, G_7$ be the bicyclic graphs on k vertices defined in Figure 3.*

(a) *Let $B_k \in \mathbf{B}_k(p,q)$:*
  (i) *if k is odd, then $t(B_k) \leq \frac{(k+1)^2}{4}$ with equality if and only if $B_k \cong G_1$;*
  (ii) *if k is even, then $t(B_k) \leq \frac{k(k+2)}{4}$ with equality if and only if $B_k \cong G_2$.*

(b) *Let $B_k \in \mathbf{B}_k(p,l,q)$:*
  (i) *if k is odd, then $t(B_k) \leq \frac{k^2-1}{4}$ with equality if and only if $B_k \cong G_3$;*
  (ii) *if k is even, then $t(B_k) \leq \frac{k^2}{4}$ with equality if and only if $B_k \cong G_4$.*

(c) *Let $B_k \in \mathbf{B}_k(P_p,P_l,P_q)$:*
  (i) *if $k = 0 \pmod 3$, then $t(B_k) \leq \frac{k(k+2)}{3}$ with equality if and only if $B_k \cong G_5$;*
  (ii) *if $k = 1 \pmod 3$, then $t(B_k) \leq \frac{k(k+2)}{3}$ with equality if and only if $B_k \cong G_6$;*

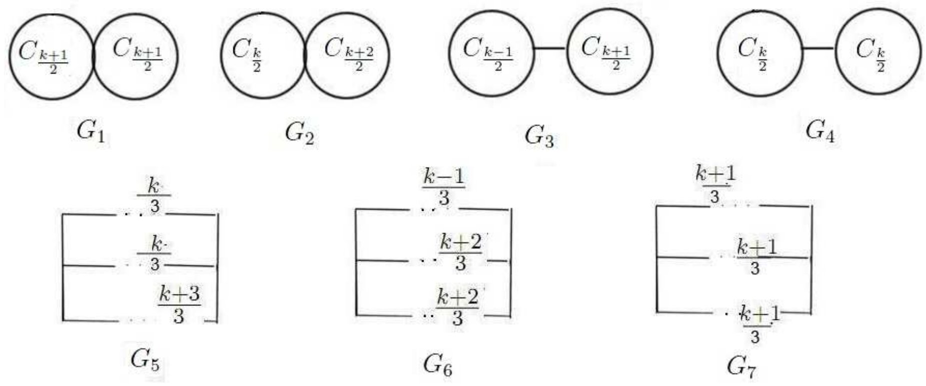

FIGURE 3. Graphs $G_1 \sim G_7$

(iii) *if* $k = 2 \pmod 3$, *then* $t(B_k) \leq \frac{(k+2)^2}{3}$ *with the equality holds if and only if* $B_k \cong G_7$.

Proof. (a) The number of spanning tree of the B_k in $\mathbf{B}_k(p, q)$ is pq. Then $t(B_k) = pq = p(k+1-p)$ and $p \in [3, \lfloor \frac{k+1}{2} \rfloor]$.
(i) If k is odd, then $t(B_k) \leq \frac{(k+1)^2}{4}$ with equality if and only if $p = q = \frac{k+1}{2}$, i.e., $B_k \cong G_1$.
(ii) If k is even, then $t(B_k) \leq \frac{k(k+2)}{4}$ with equality if and only if $p = \frac{k}{2}$ and $q = \frac{k+2}{2}$, i.e., $B_k \cong G_2$.

(b) Similarly, for $B_k \in \mathbf{B}_k(p, l, q)$, $t(B_k) = pq = p(k+1-p)$ and $p \in [3, \lfloor \frac{k}{2} \rfloor]$.
(i) If k is odd, then $t(B_k) \leq \frac{k^2-1}{4}$ with equality if and only if $l = 1, p = \frac{k-1}{2}$ and $q = \frac{k+1}{2}$, i.e., $B_k \cong G_3$.
(ii) If k is even, then $t(B_k) \leq \frac{k^2}{4}$ with equality if and only if $l = 1$ and $p = q = \frac{k}{2}$, i.e.,$B_k \cong G_4$.

(c) By the theorem in [5], for $B_k \in \mathbf{B}_k(P_p, P_l, P_q)$, we have $t(B_k) = (p+l)(q+l) - l^2 = (p+l)(k+1-p) - l^2$, where $1 \leq p \leq \lfloor \frac{k+1-l}{2} \rfloor$, $2 \leq l \leq \lfloor \frac{k}{2} \rfloor$ and $p+q+l-1 = k (k \geq 4, p \leq l \leq q)$. Let $f(x, y) = (x+y)(k+1-x) - y^2$. It is easy to see that $f(x, y)$ has a unique maximum value, which is achieved for $x = y = \frac{k+1}{3}$. If $\frac{k+1}{3}$ is not an integer, then by the linear transformation $x = \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}[Y + \frac{\sqrt{2}}{3}(k+1)], y = -\frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}[Y + \frac{\sqrt{2}}{3}(k+1)]$, we can get the function $g(X, Y) = f(x, y) = -\frac{1}{2}X^2 - \frac{3}{2}Y^2 + \frac{1}{3}(k+1)^2$ and its figure is an elliptic paraboloid. By the figure of this function, we obtain:
(i) If $k = 0 \pmod 3$, then $t(B_k) \leq \frac{k(k+2)}{3}$ with equality if and only if $p = l = \frac{k}{3}$ and $q = \frac{k+3}{3}$, i.e., $B_k \cong G_5$.
(ii) If $k = 1 \pmod 3$, then $t(B_k) \leq \frac{(k-1)(k+3)}{3}$, which is obtained when $p = l = \frac{k-1}{3}$;

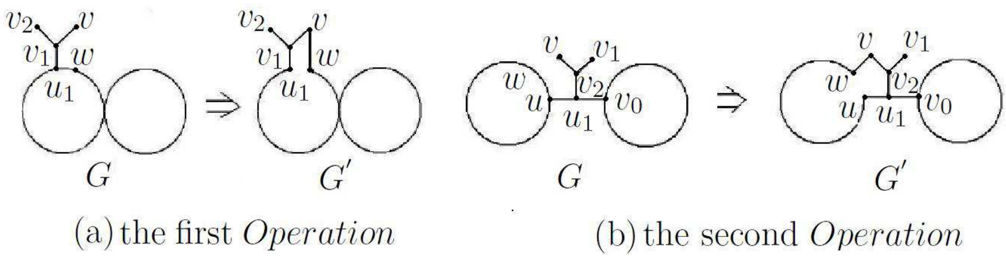

FIGURE 4. Two kinds of Operation

$t(B_k) \leq \frac{k(k+2)}{3}$, which is obtained when $p = \frac{k-1}{3}$ and $l = \frac{k+2}{3}$. Therefore $t(B_k) \leq \frac{k(k+2)}{3}$ with equality if and only if $p = \frac{k-1}{3}, l = \frac{k+2}{3}$ and $q = \frac{k+2}{3}$, i.e., $B_k \cong G_6$.

(iii) If $k = 2 \pmod 3$, then $t(B_k) \leq \frac{(k+2)^2}{3}$ with equality if and only if $p = l = q = \frac{k+1}{3}$, i.e., $B_k \cong G_7$.

Then we have the lemma. □

Now we consider any bicyclic graphs B_n, where $\delta(B_n) = 1$.
We call $H \subset G$ if H is a subgraph of G, i.e., the vertex set and the edge set of H are respectively the subset of vertex set and edge set of G. Define the following three classes of bicyclic graphs and two transformations.

Let $\mathbf{B}_n^{(1)} = \{B_n : B_k(p,q) \subset B_n, 5 \leq k < n\}$, $\mathbf{B}_n^{(2)} = \{B_n : B_k(p,l,q) \subset B_n, 6 \leq k < n\}$ and $\mathbf{B}_n^{(3)} = \{B_n : B_k(P_p, P_l, P_q) \subset B_n, 4 \leq k < n\}$.

Let $\mathbf{B}_n$ be the set of bicyclic graph with n vertices. It is easy to see that

$$\mathbf{B}_n = \left(\mathbf{B}_n^{(1)} \bigcup \mathbf{B}_n^{(2)} \bigcup \mathbf{B}_n^{(3)}\right) \bigcup \hat{\mathbf{B}}_n.$$

Suppose that G is a bicyclic graph on n vertices. Let $T(u_1)$ be a subtree of the graph G and v be a leaf of $T(u_1)$, where u_1 is the root vertex.

If vertex u_1 lies on a cycle. Let G' be the graph obtained from the graph $G \in \mathbf{B}_n^{(1)} \bigcup \mathbf{B}_n^{(2)} \bigcup \mathbf{B}_n^{(3)}$ by deleting the edge u_1w and adding a new edge wv, where w is a neighbor of u_1 on the cycle. This transformation is denoted by the first *Operation* (see Figure 4(a) for an example).

Otherwise vertex u_1 lies on the path P_{uv_0} (see Figure 4(b)) and $u_1 \notin \{u, v_0\}$. Suppose that edge uw lies on a cycle. Let G' be the graph obtained from the graph $G \in \mathbf{B_n^{(2)}}$ by deleting the edge uw and adding a new edge wv(see Figure 4(b) for an example). This transformation is denoted by the second *Operation*.

It is easy to see that $t(G) < t(G')$, and there are only two vertices with their degrees changing in the two kinds of *Operation*. Let d_v and d_{u_1} be the degree of v and u_1.

Denoted by $d(v) = d_n$ and $d(u_1) = d_{n-1}$ (in Figure 4(a)) (or $d(u) = d_{n-1}$ (In Figure 4(b))), then

$$\text{before the } Operations\text{: } \prod_{i=1}^{n} d_i = 3\prod_{i=1}^{n-2} d_i,$$
$$\text{after the } Operations\text{: } \prod_{i=1}^{n} d_i = 4\prod_{i=1}^{n-2} d_i.$$

Combining Lemma 3.1, we know $T(D[G]) < T(D[G'])$. Suppose that $G_0 \in \mathbf{B}_n$, then we can obtain $G_m \in \hat{\mathbf{B}}_n$ from the graph G_0 by using the two kinds of *Operation* in finite times. Then we have the lemma:

Lemma 3.2. *For any graph $G_0 \in \mathbf{B}_n$, there exits a graph $G_m \in \hat{\mathbf{B}}_n$ such that $T(D[G_0]) \leq T(D[G_m])$, where G_m is obtained from the graph G_0 by using the two kinds of Operation in finite times.*

The following theorem is our main result in this section.

Theorem 3.1. *Let B_n be a bicyclic graph with n vertices, where $n \geq 6$. Then $t(D[B_n]) \leq 12n(n+2)2^{3n-6}$ with equality if and only if $B_k \cong G_5$ for $k = 0 \pmod 3$, $t(D[B_n]) \leq 12n(n+2)2^{3n-6}$ with equality if and only if $B_k \cong G_6$ for $k = 1 \pmod 3$, and $t(D[B_n]) \leq 12(n+1)^2 2^{3n-6}$ with equality if and only if $B_k \cong G_7$ for $k = 2 \pmod 3$.*

Proof. By Lemmas 2.1, 3.1 and 3.2, we have following results:

For $B_n \in \mathbf{B}_n^{(1)}$, if n is odd, then $t(D[B_n]) = 4^{n-1}\prod_{i=1}^{n} d_i t(B_n) \leq 8(n+1)^2 2^{3n-6}$. Otherwise, $t(D[B_n]) \leq 8n(n+2)2^{3n-6}$.

For $B_n \in \mathbf{B}_n^{(2)}$, if n is odd, then $t(D[B_n]) \leq 9(n^2-1)2^{3n-6}$. Otherwise, $t(D[B_n]) \leq 9n^2 2^{3n-6}$.

Let $B_n \in \mathbf{B}_n^{(3)}$. If $k = 0 \pmod 3$, then $t(D[B_n]) \leq 12n(n+2)2^{3n-6}$ with equality if and only if $B_k \cong G_5$; if $k = 1 \pmod 3$, then $t(D[B_n]) \leq 12n(n+2)2^{3n-6}$ with equality if and only if $B_k \cong G_6$; if $k = 2 \pmod 3$, then $t(D[B_n]) \leq 12(n+1)^2$ with equality if and only if $B_k \cong G_7$. □

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[1] Department of Applied Mathematics, College of Science
South China Agricultural University,
Guangzhou, P. R. China, 510642
E-mail address: lwxin001@yahoo.com.cn

[2] Department of Applied Mathematics, College of Science
South China Agricultural University,
Guangzhou, P. R. China, 510642
E-mail address: weifuyi@scau.edu.cn