Some Uniqueness Results on Meromorphic Functions Sharing Two or Three Sets

Abhijit Banerjee \(^1\) and Pranab Bhattacharjee \(^2\)

Abstract. In the paper we study the uniqueness of meromorphic functions and prove some theorems which are the improvements of some results earlier given by Yi, Jank and Terglane and a recent result of the first author. Examples are provided to show that some assumptions are sharp.

1. Introduction Definitions and Results

In the paper by meromorphic function we always mean a function which is meromorphic in the open complex plane \( \mathbb{C} \). We use the standard notations and definitions of the value distribution theory available in [4]. We denote by \( T(r) \) the maximum of \( T(r, f) \) and \( T(r, g) \). The notation \( S(r) \) denotes any quantity satisfying \( S(r) = o(T(r)) \) as \( r \to \infty \), outside a possible exceptional set of finite linear measure.

If for some \( a \in \mathbb{C} \cup \{\infty\} \), \( f \) and \( g \) have the same set of \( a \)-points with same multiplicities then we say that \( f \) and \( g \) share the value \( a \) CM (counting multiplicities). If we do not take the multiplicities into account, \( f \) and \( g \) are said to share the value \( a \) IM (ignoring multiplicities).

Let \( S \) be a set of distinct elements of \( \mathbb{C} \cup \{\infty\} \) and \( E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\} \), where each zero is counted according to its multiplicity. We denote by \( \overline{E_f(S)} \) the set contains the same points as that of \( E_f(S) \) but without counting multiplicities. If

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$E_f(S) = E_g(S)$ we say that $f$ and $g$ share the set $S$ CM. On the other hand if
$\overline{E}_f(S) = \overline{E}_g(S)$, we say that $f$ and $g$ share the set $S$ IM.

Let $m$ be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_m(a; f)$
the set of all $a$-points of $f$ with multiplicities not exceeding $m$, where an $a$-point is
counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_\infty(a; f) = E_\infty(a; g)$
we say that $f$, $g$ share the value $a$ CM. For a set $S$ of distinct elements of $
\mathbb{C}$ we define $E_m(S, f) = \bigcup_{a \in S} E_m(a, f)$.

In the paper unless otherwise stated we denote by $S_1$, $S_2$ and $S_3$ the following three
sets $S_1 = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where $\omega = \cos \frac{2\pi}{n} + isin \frac{2\pi}{n}$
and $n$ is a positive integer.

Improving and extending all the previous results (c.f. [2], [3], [14]) related to the
problem of uniqueness of two meromorphic functions $f$, $g$ for which $E_f(S_i) = E_g(S_i)$,
where $i = 1, 2, 3$.

Yi [15] and independently Tohge [12] proved the following theorem.

**Theorem A.** Let $f$ and $g$ be two non-constant meromorphic functions such that
$E_f(S_i) = E_g(S_i)$, where $i = 1, 2, 3$. If $n \geq 2$ then one of the following hold:

(1.1) $f \equiv tg$,

(1.2) $f.g \equiv s$,

where $0$, $\infty$ are lacunary values of $f$ and $g$, and $s^n = 1$.


**Theorem B.** Let $f$ and $g$ be two non-constant meromorphic functions such that
$E_f(S_1) = E_g(S_1)$, $E_f(S_2) = E_g(S_2)$ and $\overline{E}_f(S_3) = \overline{E}_g(S_3)$. If $n \geq 2$ then $f$, $g$ satisfy
(1.1) or (1.2).

In 1997 H.X.Yi [17] proved the following theorems.

**Theorem C.** Let $f$ and $g$ be two non-constant meromorphic functions such that
$E_f(S_1) = E_g(S_1)$, $\overline{E}_f(S_2) = \overline{E}_g(S_2)$ and $E_f(S_3) = E_g(S_3)$. If $n \geq 2$ then $f$, $g$ satisfy
(1.1) or (1.2).

**Theorem D.** Let $f$ and $g$ be two non-constant meromorphic functions such that
$E_f(S_1) = E_g(S_1)$, $\overline{E}_f(S_2) = \overline{E}_g(S_2)$ and $\overline{E}_f(S_3) = \overline{E}_g(S_3)$. If $n \geq 3$ then $f$, $g$ satisfy
(1.1) or (1.2).
In the paper we relax the nature of sharing the sets in the above mentioned theorems.


**Theorem E.** Let $f$ and $g$ be two non-constant meromorphic functions such that $E_f(S_1) = E_g(S_1)$ and $E_f(S_3) = E_g(S_3)$. If $n \geq 6$ then then $f$, $g$ satisfy (1.1) or (1.2).

In 2001 Lahiri introduced the idea of weighted sharing of values and sets in [6], [7]. In the following definition we explain the notion.

**Definition 1.1.** [6], [7] Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

We write $f$, $g$ share $(a, k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a, k)$ then $f$, $g$ share $(a, p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a, 0)$ or $(a, \infty)$ respectively.

**Definition 1.2.** [6] Let $S$ be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and $k$ be a nonnegative integer or $\infty$. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E_f(S)} = E_f(S, 0)$.

In 2006 Lahiri and Banerjee [8] have improved Theorem E by relaxing the nature of sharing the sets with the idea of weighted sharing of values and sets which we have just discussed.

Recently the first author [1] have also investigated the problem of uniqueness of two meromorphic functions sharing the two sets $S_1$ and $S_3$ and improved and supplemented the results of Yi-Yang [19] and Lahiri-Banerjee [8].

In [1] the first author proved the following theorem.

**Theorem F.** Let $f$ and $g$ be two non-constant meromorphic functions such that $E_2(S_1, f) = E_2(S_1, g)$, $E_f(S_3, 0) = E_g(S_3, 0)$ and $n \geq 8$ then $f$, $g$ satisfy (1.1) or (1.2).

In this paper we shall improve Theorem F by showing that the assumption $n \geq 8$ can be replaced by $n \geq 7$. 
Following theorems are the main results of the paper.

**Theorem 1.1.** If $E_m(S_1, f) = E_m(S_1, g)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, k) = E_g(S_3, k)$, where $k(2m - 17) > 12$ and $n \geq 2$ then $f, g$ satisfy one of (1.1) or (1.2).

**Theorem 1.2.** If $E_m(S_1, f) = E_m(S_1, g)$, $E_f(S_2, p) = E_g(S_2, p)$, $E_f(S_3, 0) = E_g(S_3, 0)$, where $p(2m - 17) > 12$ and $n \geq 2$ then $f, g$ satisfy one of (1.1) or (1.2).

**Theorem 1.3.** If $E_7(S_1, f) = E_7(S_1, g)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, 0) = E_g(S_3, 0)$ and $n \geq 3$ then $f, g$ satisfy one of (1.1) or (1.2).

**Theorem 1.4.** If $E_2(S_1, f) = E_2(S_1, g)$, $E_f(S_3, 0) = E_g(S_3, 0)$ and $n \geq 7$ then $f, g$ satisfy (1.1) or (1.2).

**Example 1.1.** Let $f(z) = (1 - e^z)^3$ and $g(z) = 3(e^{-z} - e^{-2z})$ and $S_1 = \{1\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$. Clearly $f$ and $g$ share $(S_1, \infty)$, $(S_2, 0)$, $(S_3, \infty)$, but neither condition (1.1) nor (1.2) is satisfied. So the condition $n \geq 2$ in Theorem 1.1 is the best possible.

**Example 1.2.** Let $f(z) = \frac{(1 - 3e^z)}{(1 - e^z)^3}$ and $g(z) = \frac{(1 - 3e^z)}{4(1 - e^z)}$ and and $S_1$, $S_2$, $S_3$ be same as defined in Example 1.1. Clearly $f$ and $g$ share $(S_1, \infty)$, $(S_2, \infty)$, $(S_3, 0)$, but neither condition (1.1) nor (1.2) is satisfied. So the condition $n \geq 2$ in Theorem 1.2 is the best possible.

**Example 1.3.** Let $f(z) = \frac{(e^{2z} + 1)^2}{2e^z(e^{2z} - 1)}$ and $g(z) = \frac{2ie^z(e^{2z} + 1)}{(e^{2z} - 1)^2}$ and $S_1 = \{-1, 1\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$. Clearly $f$ and $g$ share $(S_1, \infty)$, $(S_2, 0)$, $(S_3, 0)$, but neither condition (1.1) nor (1.2) is satisfied. So the condition $n \geq 3$ in Theorem 1.3 is the best possible.

**Corollary 1.1.** When $k = \infty$ and $p = \infty$ both Theorem 1.1, Theorem 1.2 hold for $m \geq 9$.

We explain some definitions and notations which are used in the paper.

**Definition 1.3.** [5] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid = 1)$ the counting function of simple $a$-points of $f$. For a positive integer $m$ we denote by $N(r, a; f \mid \leq m)(N(r, a; f \mid \geq m))$ the counting function of those $a$-points of $f$ whose multiplicities are not greater (less) than $m$ where each $a$-point is counted according to its multiplicity.
\[ \mathcal{N}(r, a; f \mid \leq m)(\mathcal{N}(r, a; f \mid \geq m)) \] are defined similarly, where in counting the \( a \)-points of \( f \) we ignore the multiplicities.

Also \( N(r, a; f \mid < m), N(r, a; f \mid > m), \mathcal{N}(r, a; f \mid < m) \) and \( \mathcal{N}(r, a; f \mid > m) \) are defined analogously.

**Definition 1.4.** We denote by \( \mathcal{N}(r, a; f \mid = k) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicity is exactly \( k \), where \( k \geq 2 \) is an integer.

**Definition 1.5.** Let \( f \) and \( g \) be two non-constant meromorphic functions such that \( f \) and \( g \) share a value \( a \) IM where \( a \in \mathbb{C} \cup \{\infty\} \). Let \( z_0 \) be an \( a \)-point of \( f \) with multiplicity \( p \), an \( a \)-point of \( g \) with multiplicity \( q \). We denote by \( \mathcal{N}_L(r, a; f)(\mathcal{N}_L(r, a; g)) \) the counting function of those \( a \)-points of \( f \) and \( g \) where \( p > q \) \((q > p)\), each \( a \)-point is counted only once.

**Definition 1.6.** Let \( f \) and \( g \) be two non-constant meromorphic functions and \( m \) be a positive integer such that \( E_m(a; f) = E_m(a; g) \) where \( a \in \mathbb{C} \cup \{\infty\} \). Let \( z_0 \) be an \( a \)-point of \( f \) with multiplicity \( p > 0 \), an \( a \)-point of \( g \) with multiplicity \( q > 0 \). We denote by \( \mathcal{N}_L^{m+1}(r, a; f)(\mathcal{N}_L^{m+1}(r, a; g)) \) the counting function of those common \( a \)-points of \( f \) and \( g \) where \( p > q \) \((q > p)\), each \( a \)-point is counted only once.

**Definition 1.7.** Let \( z_0 \) be a \( 1 \)-point of \( f \) with multiplicity \( p \), a \( 1 \)-point of \( g \) with multiplicity \( q \). We denote by \( \mathcal{N}_E^{m+1}(r, 1; f) \) the counting function of those \( 1 \)-points of \( f \) and \( g \) where \( p = q \geq m + 1 \), each point in this counting function is counted only once.

**Definition 1.8.** \([7]\) We denote by \( N_2(r, a; f) = \mathcal{N}(r, a; f) + \mathcal{N}(r, a; f \mid \geq 2) \).

**Definition 1.9.** Let \( m \) be a positive integer. Also let \( z_0 \) be a zero of \( f(z) - a \) of multiplicity \( p \) and a zero of \( g(z) - a \) of multiplicity \( q \). We denote by \( \mathcal{N}_{f \geq m+1}(r, a; f \mid g \neq a)(\mathcal{N}_{g \geq m+1}(r, a; g \mid f \neq a)) \) the reduced counting functions of those \( a \)-points of \( f \) and \( g \) for which \( p \geq m + 1 \) and \( q = 0 \) \((q \geq m + 1 \text{ and } p = 0)\).

**Definition 1.10.** \([6],[7]\) Let \( f, g \) share \((a, 0)\). We denote by \( \mathcal{N}_*(r, a; f, g) \) the reduced counting function of those \( a \)-points of \( f \) whose multiplicities differ from the multiplicities of the corresponding \( a \)-points of \( g \).

Clearly \( \mathcal{N}_*(r, a; f, g) \equiv \mathcal{N}_*(r, a; g, f) \) and \( \mathcal{N}_*(r, a; f, g) = \mathcal{N}_L(r, a; f) + \mathcal{N}_L(r, a; g) \).
Definition 1.11. For $E_m(a; f) = E_m(a; g)$ we define $N_\otimes(r, a; f, g)$ as follows

$$N_\otimes(r, a; f, g) = N_{m+1}^L(r, a; f) + N_{m+1}^L(r, a; g) + N_{f \geq m+1}(r, a; f | g \neq a) + N_{g \geq m+1}(r, a; g | f \neq a)$$

$$\leq N(r, a; f | \geq m + 1) + N(r, a; g | \geq m + 1).$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let $F$ and $G$ be two non-constant meromorphic functions defined in $C$. Henceforth we shall denote by $H$, $U$ and $V$ the following three functions.

$$H = \left( \frac{F'}{F''} - \frac{2F'}{F - 1} \right) - \left( \frac{G'}{G''} - \frac{2G'}{G - 1} \right),$$

$$U = \frac{F'}{F - 1} - \frac{G'}{G - 1}$$

and

$$V = \frac{F'}{F - 1} - \frac{F'}{F} - \left( \frac{G'}{G - 1} - \frac{G'}{G} \right) = \frac{F'}{F(F - 1)} - \frac{G'}{G(G - 1)}.$$

Lemma 2.1. [11] For $E_m(1; F) = E_m(1; G)$ and $H \neq 0$ then

$$N(r, 1; F | = 1) = N(r, 1; G | = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2. [9] If $N(r, 0; f^{(k)} | f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity then

$$N(r, 0; f^{(k)} | f \neq 0) \leq kN(r, \infty; f) + N(r, 0; f | < k) + kN(r, 0; f | \geq k) + S(r, f).$$

Lemma 2.3. Let $E_m(1; f) = E_m(1; g)$ and $3 \leq m < \infty$. Then

$$N(r, 1; f | = 2) + 2N(r, 1; f | = 3) + \ldots + N(r, 1; f | = m)$$

$$+ mN_{E}^{(m+1)}(r, 1; f) + mN_{L}^{(m+1)}(r, 1; f) + (m + 1)N_{L}^{(m+1)}(r, 1; g)$$

$$+ mN_{g \geq m+1}(r, 1; g | f \neq 1)$$

$$\leq N(r, 1; g) - N(r, 1; g).$$
Proof. Since $E_{m_1}(1; f) = E_{m_1}(1; g)$, we note that common zeros of $f - 1$ and $g - 1$ up to multiplicity $m$ are same. Let $z_0$ be a 1-point of $f$ with multiplicity $p$ and a 1-point of $g$ with multiplicity $q$. If $q = m + 1$ the possible values of $p$ are as follows (i) $p = m + 1$ (ii) $p \geq m + 2$ (iii) $p = 0$. Similarly when $q = m + 2$ the possible values of $p$ are (i) $p = m + 1$ (ii) $p = m + 2$ (iii) $p \geq m + 3$ (iv) $p = 0$. If $q \geq m + 3$ we can similarly find the possible values of $p$. Now the lemma follows from above explanation.

Lemma 2.4. Let $E_2(1; f) = E_2(1; g)$. Then

$$N(r, 1; f | \geq 2) + 2\overline{N}_E(r, 1; f) + 2\overline{N}_L(r, 1; f) + 2\overline{N}_g(r, 1; g) + 2\overline{N}_{g \geq 3}(r, 1; g | f \neq 1) \leq N(r, 1; g) - \overline{N}(r, 1; g).$$

Proof. Since $E_2(1; f) = E_2(1; g)$, we note that the simple and double 1-points of $f$ and $g$ are same. Let $z_0$ be a 1-point of $f$ with multiplicity $p$ and a 1-point of $g$ with multiplicity $q$. If $q = 3$ the possible values of $p$ are as follows (i) $p = 3$ (ii) $p \geq 4$ (iii) $p = 0$. Similarly when $q = 4$ the possible values of $p$ are (i) $p = 3$ (ii) $p = 4$ (iii) $p \geq 5$ (iv) $p = 0$. If $q \geq 5$ we can similarly find the possible values of $p$. Now the lemma follows from above explanation.

Lemma 2.5. Let $E_{m_1}(1; F) = E_{m_1}(1; G)$ and $F$, $G$ share $(\infty; 0)$. Also let $H \neq 0$. Then

$$N(r, \infty; H) \leq \overline{N}(r, 0; F | \geq 2) + \overline{N}(r, 0; G | \geq 2) + \overline{N}_v(r, \infty; F, G)$$
$$+ \overline{N}_e(r, 1; F, G) + \overline{N}_o(r, 0; F') + \overline{N}_o(r, 0; G'),$$

where $\overline{N}_o(r, 0; F')$ is the reduced counting function of those zeros of $F'$ which are not the zeros of $F(F' - 1)$ and $\overline{N}_o(r, 0; G')$ is similarly defined.

Proof. The proof of the lemma can be carried out in the line of the proof of Lemma 4 [8]. So we omit it.

Lemma 2.6. Let $E_{m_1}(1; F) = E_{m_1}(1; G)$ and $F$, $G$ share $(0, p)$ and $(\infty; k)$. Also let $H \neq 0$. Then

$$N(r, \infty; H) \leq \overline{N}_v(r, 0; F; G) + \overline{N}_v(r, \infty; F, G) + \overline{N}_e(r, 1; F, G)$$
$$+ \overline{N}_o(r, 0; F') + \overline{N}_o(r, 0; G').$$

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.5.
Henceforth we assume
\[(2.1) \quad F = f^n \quad \text{and} \quad G = g^n.\]

**Lemma 2.7.** Let \( F, G \) be given by (2.1) and \( H \neq 0 \). If \( E_{m_3}(1; F) = E_{m_3}(1; G) \), \( f, g \) share \((\infty, k), (0, p)\), where \( 3 \leq m < \infty \). Then
\[
\begin{align*}
\quad nT(r, f) \leq & \quad N(r, 0; f) + N(r, \infty; f) + N(r, 0; g) + N(r, \infty; g) + N_s(r, 0; f, g) \\
+ & \quad (m - 2)N(r, 1; F = m) - (m - 2)N_{L}^{m+1}(r, 1; F) \\
+ & \quad (m - 1)N_{E}^{m+1}(r, 1; G) - (m - 1)N_{L}^{m+1}(r, 1; F) \\
+ & \quad 2N_{F \geq m+1}(r, 1; F \mid G \neq 1) - (m - 1)N_{G \geq m+1}(r, 1; G \mid F \neq 1) \\
+ & \quad S(r, f) + S(r, g).
\end{align*}
\]

*Similar expressions also hold for \( g \).*

**Proof.** By the second fundamental theorem we get
\[(2.2) \quad T(r, F) + T(r, G) \leq N(r, 0; F) + N(r, \infty; F) + N(r, 0; G) + N(r, \infty; G) \]
\[
+ \quad N(r, 1; F) + N(r, 1; G) - N_0(r, 0; F') - N_0(r, 0; G') \\
+ \quad S(r, F) + S(r, G).
\]

Using Lemmas 2.1, 2.3 and 2.6 we see that
\[(2.3) \quad N(r, 1; F) + N(r, 1; G) \leq N(r, 1; F | = 1) + N(r, 1; F | = 2) + \ldots + N(r, 1; F | = m) \]
\[
+ \quad N_{E}^{m+1}(r, 1; F) + N_{L}^{m+1}(r, 1; F) + N_{L}^{m+1}(r, 1; G) + N_{F \geq m+1}(r, 1; F \mid G \neq 1) \]
\[
+ \quad N(r, 1; G) \leq N_s(r, 0; f, g) + N_s(r, \infty; f, g) + N_s(r, 1; F, G) + N(r, 1; F | = 2) + \ldots \\
+ \quad N(r, 1; F | = m) + N_{E}^{m+1}(r, 1; F) + N_{L}^{m+1}(r, 1; F) + N_{L}^{m+1}(r, 1; G) \\
+ \quad N_{F \geq m+1}(r, 1; F \mid G \neq 1) + T(r, G) - m(r, 1; G) + O(1) - N(r, 1; F | = 2) \\
- \quad 2N(r, 1; F | = 3) - (m - 1)N(r, 1; F | = m) - \ldots - mN_{E}^{m+1}(r, 1; F) \\
- \quad mN_{L}^{m+1}(r, 1; F) - (m + 1)N_{L}^{m+1}(r, 1; G) - mN_{G \geq m+1}(r, 1; G \mid F \neq 1) \\
+ \quad N_0(r, 0; F') + N_0(r, 0; G') + S(r, F) + S(r, G)
\]
\[ \begin{align*}
& \leq \sum_{k=0}^{n} (n-1)N(r, 0; f, g) + \sum_{k=0}^{m} (2n-1)N(r, 0; f, g) + T(r, G) - m(r, 1; G) - N(r, 1; F | \equiv 3) \\
& \quad - 2N(r, 1; F | \equiv 4) - \ldots - (m-2)N(r, 1; F | \equiv m) - (m-2)N^{(m+1)}_L(r, 1; F) \\
& \quad - (m-1)N^{(m+1)}_L(r, 1; G) - (m-1)N^{(m+1)}_E(r, 1; F) \\
& \quad - (m-1)N_{G \geq m+1}(r, 1; G | F \neq 1) + 2N_{F \geq m+1}(r, 1; F | G \neq 1) \\
& \quad + N_0(r, 0; F^*) + N_0(r, 0; G^*) + S(r, F) + S(r, G).
\end{align*} \]

Using (2.3) in (2.2), the lemma follows.

**Lemma 2.8.** Let \( F, G \) be given by (2.1) and \( H \neq 0 \). If \( E_2(1; F) = E_2(1; G) \), \( f, g \) share \((\infty, 0)\). Then
\[
\begin{align*}
nT(r, f) \\
& \leq N_2(r, 0; F) + N(r, \infty; f) + N_2(r, 0; G) + N(r, \infty; g) + N_\ast(r, \infty; f, g) \\
& \quad - m(r, 1; G) - N^{(3)}_E(r, 1; F) + 2N_{F \geq 3}(r, 1; F | G \neq 1) - N_{G \geq 3}(r, 1; G | F \neq 1) \\
& \quad + S(r, f) + S(r, g).
\end{align*}
\]

Similar expressions also hold for \( g \).

**Proof.** We omit the proof since using Lemmas 2.1, 2.4 and 2.5 the proof can be carried out in the line of proof of Lemma 2.7.

**Lemma 2.9.** [17] Let \( F, G \) be given by (2.2). If \( F, G \) share \((0, 0)\) and \( U \equiv 0 \) then \( F \equiv G \).

**Lemma 2.10.** [17] Let \( F, G \) be given by (2.2). If \( F, G \) share \((\infty, 0)\) and \( V \equiv 0 \) then \( F \equiv G \).

**Lemma 2.11.** Let \( F, G \) be given by (2.1) and \( F \neq G \). If \( E_m(S_1; f) = E_m(S_1; g) \), \( E_f(S_2, p) = E_g(S_2, p) \) and \( E_f(S_3, k) = E_g(S_3, k) \), where \( 1 \leq m \leq \infty \), \( 0 \leq p < \infty \), \( 0 \leq k < \infty \) then
\[
\begin{align*}
& \left( \frac{1}{nk + n - 1} \right) N(r, 0; f \mid 1) + (2n-1)N(r, 0; f \mid 2) + \ldots \\
& \quad + \left( np + n - 1 - \frac{1}{nk + n - 1} \right) N(r, 0; f \mid \geq p + 1) \\
& \leq \frac{nk + n}{nk + n - 1} N_\ast(r, 1; F, G) + S(r).
\end{align*}
\]

**Proof.** Since \( F \neq G \) we have from Lemmas 2.9 and 2.10 that \( U \neq 0 \) and \( V \neq 0 \). According to the statement of the lemma it is clear that \( E_m(1; F) = E_m(1; G) \)
and $F$, $G$ share $(0; np)$, $(\infty; nk)$ and so a zero (pole) of $F$ with multiplicity $r \geq np + 1(\geq nk + 1)$ is a zero (pole) of $G$ with multiplicity $s \geq np + 1(\geq nk + 1)$ and vice versa. We note that $F$ and $G$ have no zero (pole) of multiplicity $q$ where $np < q < np + n(nk < q < nk + n)$. Hence we get from the definition of $U$

\[(2.4) \quad (n-1)N(r,0; f \mid= 1) + (2n-1)N(r,0; f \mid= 2) + \ldots \]
\[\quad + (np + n - 1)N(r,0; f \mid\geq p + 1)\]
\[= (n-1)\overline{N}(r,0; F \mid= n) + (2n-1)\overline{N}(r,0; F \mid= 2n) + \ldots \]
\[\quad + (np + n - 1)\overline{N}(r,0; F \mid\geq np + n)\]
\[\leq N(r,0; U)\]
\[\leq T(r, U) + O(1)\]
\[\leq N(r, \infty; U) + S(r)\]
\[\leq \overline{N}(r, \infty; F, G) + \overline{N}_\circ(r,1; F, G) + S(r)\]
\[\leq \overline{N}(r, \infty; F \mid\geq nk + n) + \overline{N}_\circ(r,1; F, G) + S(r)\].

In a similar argument as above we get from the definition of $V$

\[(2.5) \quad (nk + n - 1)\overline{N}(r, \infty; F \mid\geq nk + n)\]
\[\leq N(r,0; V)\]
\[\leq N(r, \infty; V) + S(r)\]
\[\leq \overline{N}(r,0; F \mid\geq np + n) + \overline{N}_\circ(r,1; F, G) + S(r)\].

Using (2.5) in (2.4) the lemma follows. \hfill \Box

**Lemma 2.12.** Let $F$, $G$ be given by (2.1) and $F \neq G$. If $E_m(S_1; f) = E_m(S_1; g)$, $E_f(S_2,p) = E_g(S_2,p)$ and $E_f(S_3,k) = E_g(S_3,k)$, where $1 \leq m < \infty$, $0 \leq p < \infty$, $0 \leq k < \infty$ then

\[(n-1)\overline{N}(r, \infty; f \mid= 1) + (2n-1)\overline{N}(r, \infty; f \mid= 2) + \ldots \]
\[+ \left(\frac{nk + n - 1}{np + n - 1} - 1\right)\overline{N}(r, \infty; f \mid\geq k + 1)\]
\[\leq \frac{np+n}{np+n-1} \overline{N}_\circ(r,1; F, G) + S(r)\].

**Proof.** We omit the proof since it can be carried out in the line of proof of Lemma 2.11. \hfill \Box
Lemma 2.13. [1] Let $F, G$ be given by (2.1) and $V \not\equiv 0$. If $f, g$ share $(\infty, k)$, where $0 \leq k < \infty$, and $E_m(1; F) = E_m(1; G)$, then
\[
(nk + n - 1) \bar{N}(r, \infty; f \mid \geq k + 1) = (nk + n - 1)\bar{N}(r, \infty; F \mid \geq nk + n)
\leq \frac{m + 1}{m}[\bar{N}(r, 0; f) + \bar{N}(r, 0; g)]
+ \frac{2}{m} \bar{N}(r, \infty; f) + S(r, f) + S(r, g).
\]

Lemma 2.14. [16] If $H \equiv 0$ then $T(r, G) = T(r, F) + O(1)$. Also if $H \equiv 0$ and
\[
\limsup_{r \to \infty, r \not\in I} \frac{\bar{N}(r, 0; F) + \bar{N}(r, \infty; F) + \bar{N}(r, 0; G) + \bar{N}(r, \infty; G)}{T(r, F)} < 1
\]
where $I \subset (0, 1)$ is a set of infinite linear measure, then $F \equiv G$ or $F.G \equiv 1$.

Lemma 2.15. [18] If $H \equiv 0$, then $F, G$ share $(1, \infty)$. If further $F, G$ share $(\infty, 0)$ then $F, G$ share $(\infty, \infty)$.

Lemma 2.16. Let $F, G$ be given by (2.1) and $n \geq 2$. Also let $E_m(1; F) = E_m(1; G)$. If $f, g$ share $(0, 0), (\infty, k)$, where $0 \leq k < \infty$ and $H \equiv 0$. Then $f, g$ satisfy one of (1.1) or (1.2).

Proof. Since $H \equiv 0$ we get from Lemma 2.15 that $F$ and $G$ share $(1, \infty)$ and $(\infty, \infty)$. So $\bar{N}(r, 1; F, G) = \bar{N}_*(r, \infty; F, G) \equiv 0$. If possible let us suppose (1.1) is not satisfied. Then clearly $F \not\equiv G$. Since $F \not\equiv G$ we have from Lemmas 2.9 and 2.10 respectively $U \not\equiv 0$ and $V \not\equiv 0$. Hence
\[
(n - 1)\bar{N}(r, 0; f) = (n - 1)\bar{N}(r, 0; g)
\leq N(r, 0; U)
\leq N(r, \infty; U) + S(r)
\leq \bar{N}_*(r, \infty; F, G) + \bar{N}_*(r, 1; F, G) + S(r)
= S(r).
\]
and
\[
(n - 1)\bar{N}(r, \infty; f) = (n - 1)\bar{N}(r, \infty; g)
\leq N(r, 0; V)
\leq N(r, \infty; V) + S(r)
\leq \bar{N}_*(r, 0; f, g) + \bar{N}_*(r, 1; F, G) + S(r).
= S(r).
\]
Since \( n \geq 2 \) we have from above \( \mathcal{N}(r, 0; f) = \mathcal{N}(r, 0; g) = S(r) \) and \( \mathcal{N}(r, \infty; f) = \mathcal{N}(r, \infty; g) = S(r) \). Hence using Lemma 2.14 we get the conclusion of the lemma. \( \square \)

**Lemma 2.17.** Let \( F, G \) be given by (2.1), \( E_m(1; F) = E_m(1; G) \), \( 1 \leq m < \infty \). Then

\[
\begin{align*}
(\text{i}) \quad & \mathcal{N}(r, 1; F | \geq m + 1) \leq \frac{1}{m} \left[ \mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; f) - N_\oplus(r, 0; f') \right] + S(r, f), \\
(\text{ii}) \quad & \mathcal{N}(r, 1; G | \geq m + 1) \leq \frac{1}{m} \left[ \mathcal{N}(r, 0; g) + \mathcal{N}(r, \infty; g) - N_\oplus(r, 0; g') \right] + S(r, g),
\end{align*}
\]

where \( N_\oplus(r, 0; f') = N(r, 0; f' | f \neq 0, \omega_1, \omega_2 \ldots \omega_n) \).

**Proof.** We prove only (i).

Using Lemma 2.2 we see that

\[
\begin{align*}
\mathcal{N}(r, 1; F | \geq m + 1) & \leq \frac{1}{m} \left( N(r, 1; F) - \mathcal{N}(r, 1; F) \right) \\
& \leq \frac{1}{m} \left[ \sum_{j=1}^{n} \left( N(r, \omega_j; f) - \mathcal{N}(r, \omega_j; f) \right) \right] \\
& \leq \frac{1}{m} \left( N(r, 0; f' | f \neq 0) - N_\oplus(r, 0; f') \right) \\
& \leq \frac{1}{m} \left[ \mathcal{N}(r, 0; f) + \mathcal{N}(r, \infty; f) - N_\oplus(r, 0; f') \right] + S(r, f).
\end{align*}
\]

This proves the lemma. \( \square \)

### 3. Proofs of the theorems

**Proof of Theorem 1.1.** Let \( F, G \) be given by (2.1). Then \( E_m(1; F) = E_m(1; G) \) and \( f, g \) share \((0, 0)\) and \((\infty; k)\). We consider the following cases.

**Case 1.** Let \( H \not\equiv 0 \). Then \( F \not\equiv G \). Noting that \( f \) and \( g \) share \((0, 0)\) and \((\infty; k)\) implies \( \mathcal{N}_*(r, 0; f, g) \leq \mathcal{N}(r, 0; f) = \mathcal{N}(r, 0; g) \) and \( \mathcal{N}_*(r, \infty; f, g) \leq \mathcal{N}(r, \infty; f | \geq k + 1) = \mathcal{N}(r, \infty; g | \geq k + 1) \), using Lemma 2.7, Lemma 2.11 with \( p = 0 \), (2.5) with \( k = 0 \) and \( p = 0 \) and Lemma 2.12 with \( p = 0 \) and Lemma 2.17 we obtain

\[
\begin{align*}
(3.1) \quad & nT(r, f) + nT(r, g) \\
& \leq 3 \mathcal{N}(r, 0; f) + 3 \mathcal{N}(r, 0; g) + 2 \mathcal{N}(r, \infty; f) + 2 \mathcal{N}(r, \infty; g) \\
& + 2 \mathcal{N}_*(r, \infty; f, g) - (m - 3) \mathcal{N}_\oplus(r, 1; F, G) + S(r, f) + S(r, g)
\end{align*}
\]
Proof of Theorem 1.2. Let Case 2. □

From (3.1) we see that

$$\left( n + 2 - \frac{6}{m} - \frac{8}{m(n-1)} - \frac{n(12k + 16) + \frac{8n(k+1)}{n-1}}{m(n-1)(nk + n - 1) - m} \right) T(r, f)$$
$$+ \left( n + 2 - \frac{6}{m} - \frac{8}{m(n-1)} - \frac{n(12k + 16) + \frac{8n(k+1)}{n-1}}{m(n-1)(nk + n - 1) - m} \right) T(r, g)$$

From (3.1) we see that

$$(3.2) \quad \left( n + 2 - \frac{6}{m} - \frac{8}{m(n-1)} - \frac{n(12k + 16) + \frac{8n(k+1)}{n-1}}{m(n-1)(nk + n - 1) - m} \right) T(r, f)$$
$$+ \left( n + 2 - \frac{6}{m} - \frac{8}{m(n-1)} - \frac{n(12k + 16) + \frac{8n(k+1)}{n-1}}{m(n-1)(nk + n - 1) - m} \right) T(r, g)$$

Since $n \geq 2$ and $k(2m - 17) > 12$, (3.2) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemma 2.16. □

Proof of Theorem 1.2. We omit the proof since it can be carried out in the line of proof of Theorem 1.1.

Proof of Theorem 1.3. Let $F, G$ be given by (2.1). Then $E_7(1; F) = E_7(1; G)$ and $f, g$ share $(0,0)$ and $(\infty; 0)$. We consider the following cases.

Case 1. Let $H \not= 0$. Then $F \not= G$. Noting that $f$ and $g$ share $(0,0)$ and $(\infty; 0)$ implies

$$\overline{N}_s(r, 0; f, g) \leq \overline{N}(r, 0; f) = \overline{N}(r, 0; g)$$
and $\overline{N}_s(r, \infty; f, g) \leq \overline{N}(r, \infty; f) = \overline{N}(r, \infty; g)$. 

$\overline{N}_s(r, 0; f, g) \leq \overline{N}(r, 0; f) = \overline{N}(r, 0; g)$ and $\overline{N}_s(r, \infty; f, g) \leq \overline{N}(r, \infty; f) = \overline{N}(r, \infty; g)$,
using Lemma 2.7, Lemmas 2.11 and 2.12 with \( p = 0, \ k = 0 \) and Lemma 2.17 we obtain

\[
(3.3) \quad nT(r, f) + nT(r, g) \\
\leq 3 \ \mathcal{N}(r, 0; f) + 3 \ \mathcal{N}(r, 0; g) + 3 \ \mathcal{N}(r, \infty; f) + 3 \ \mathcal{N}(r, \infty; g) \\
-4 \ \mathcal{N}_\otimes(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq \frac{6}{n-2} \ \mathcal{N}_\otimes(r, 1; F, G) + \frac{6}{n-2} \ \mathcal{N}_\otimes(r, 1; F, G) - 4 \ \mathcal{N}_\otimes(r, 1; F, G) \\
+ S(r, f) + S(r, g) \\
\leq \left( \frac{12}{(n-2)} - 4 \right) \ \mathcal{N}_\otimes(r, 1; F, G) + S(r, f) + S(r, g) \\
\leq \frac{2}{7} \left( \frac{20 - 4n}{n-2} \right) T(r, f) + \frac{2}{7} \left( \frac{20 - 4n}{n-2} \right) T(r, g) \\
+ S(r, f) + S(r, g).
\]

From (3.3) we see that

\[
(3.4) \quad \left( n - \frac{40 - 8n}{7(n-2)} \right) T(r, f) + \left( n - \frac{40 - 8n}{7(n-2)} \right) T(r, g) \leq S(r, f) + S(r, g).
\]

Since \( n \geq 3 \), (3.4) leads to a contradiction.

**Case 2.** Let \( H \equiv 0 \). Then the theorem follows from Lemma 2.16. \( \square \)

**Proof of Theorem 1.4.** Let \( F, G \) be given by (2.1). Then \( E_{2b}(1; F) = E_{2b}(1; G) \) and \( f, g \) share \((\infty; 0)\). We consider the following cases.

**Case 1.** Let \( H \not\equiv 0 \). Then \( F \not\equiv G \). So from Lemma 2.10 we get \( V \not\equiv 0 \). Hence using Lemmas 2.8, 2.13 with \( m = 2 \) and \( k = 0 \) and Lemma 2.17 we obtain

\[
(3.5) \quad nT(r, f) + nT(r, g) \\
\leq 4 \ \mathcal{N}(r, 0; f) + 4 \ \mathcal{N}(r, 0; g) + 2 \ \mathcal{N}(r, \infty; f) + 2 \ \mathcal{N}(r, \infty; g) \\
+ 2 \ \mathcal{N}_*(r, \infty; f, g) + \mathcal{N}_{F \geq 3}(r, 1; F \mid G \neq 1) \\
+ \mathcal{N}_{G \geq 3}(r, 1; G \mid F \neq 1) + S(r, f) + S(r, g) \\
\leq 4 \ \mathcal{N}(r, 0; f) + 4 \ \mathcal{N}(r, 0; g) + \frac{1}{2} \left( \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g) \right) \\
+ 7 \ \mathcal{N}(r, \infty; f) + S(r, f) + S(r, g) \\
\leq \left( \frac{9}{2} + \frac{21}{2(n-2)} \right) \left\{ \mathcal{N}(r, 0; f) + \mathcal{N}(r, 0; g) \right\} + S(r, f) + S(r, g).
\]
From (3.5) we see that

\[(n - \frac{9}{2} - \frac{21}{2(n-2)}) \, T(r, f) + (n - \frac{9}{2} - \frac{21}{2(n-2)}) \, T(r, g) \leq S(r, f) + S(r, g)\]

Since \(n \geq 7\), (3.6) leads to a contradiction.

**Case 2.** Let \(H \equiv 0\). Since \(n \geq 7\) from Lemma 2.14 it follows that \(f\) and \(g\) satisfy one of (1.1) or (1.2).

\[\square\]

**References**

1 Department of Mathematics,  
West Bengal State University,  
Barasat, North 24 Prgs.,  
West Bengal 700126, India.  
E-mail address: abanerjee_kal@yahoo.co.in, abanerjee_kal@rediffmail.com

2 Department of Mathematics,  
Hooghly Mohsin College,  
Chinsurah, Hooghly,  
West Bengal 712101, India.