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INVARIANT STRUCTURES ON THE 6–DIMENSIONAL GENERALIZED HEISENBERG GROUP

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Abstract. In this paper, using the theory of canonical structures on homogeneous k–symmetric spaces, we construct four left-invariant metric f–structures on the 6–dimensional generalized Heisenberg group. It provides new invariant examples for the classes of nearly Kähler and Hermitian f–structures as well as almost Hermitian G_1–structures.

1. Introduction

An important role among homogeneous manifolds of Lie groups is occupied by *homogeneous k-symmetric spaces*, that is, the homogeneous spaces generated by Lie groups automorphisms Φ of order k ($\Phi^k = id$) [17]. The remarkable feature of these spaces is that any homogeneous k-symmetric space $(G/H, \Phi)$ admits a natural associated object, the commutative algebra $\mathcal{A}(\theta)$ [4] of *canonical affinor structures*. This algebra contains well-known classical structures, such as almost complex structures, almost product structures, f–structures of K. Yano ($f^3 + f = 0$), h–structures ($h^3 - h = 0$) (see [4], [8]). It should be noted that certain homogeneous manifolds G/H can be generated by different automorphisms of the Lie group G. This fact implies a good opportunity to construct different invariant canonical structures on the same underlying G/H. In particular, some Lie groups can be considered as homogeneous k–symmetric spaces for different orders $k \geq 2$ (see, for example, [17], [21]).

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In the paper, we apply this idea to construct different f–structures and almost complex structures on the 6–dimensional generalized Heisenberg group (N, g). It should be mentioned that the group (N, g) possesses many remarkable properties (see for details [14], [15], [21] and others). Specifically, this group can be simultaneously represented as homogeneous k–symmetric spaces for $k = 3, 4$, and 6. We concentrate on the four left-invariant metric canonical f–structures (two of them are almost Hermitian structures) on the Riemannian homogeneous 6–symmetric space (N, g). Two of these structures were investigated before in [21], [5], [6], the other two structures we study here. In particular, we prove that the canonical f–structure f_2 is a non-integrable nearly Kähler and Hermitian f–structure. Besides, the almost Hermitian structure J belongs to the class $\mathbf{G_1}$ (see [12]).

The paper is organized as follows.

In Section 2, we collect preliminary basic notions on almost Hermitian structures and metric f–structures on manifolds. In particular, the description of the main classes of metric f–structures is included. The inclusive relations between these classes are also indicated and the special case of almost Hermitian structures is considered.

In Section 3, we give a brief exposition of canonical structures on homogeneous regular Φ-spaces. The most important particular case of canonical f–structures on homogeneous k–symmetric spaces is presented in more detail. For future consideration, we indicate the precise formulae for all the canonical f–structures for homogeneous k–symmetric spaces of orders $k = 3, 4$, and 6.

Finally, in Section 4, we examine in detail special left-invariant f–structures on the 6–dimensional generalized Heisenberg group (N, g). More exactly, we represent this group as a Riemannian homogeneous 6–symmetric space and construct all the canonical f–structures for the case. When investigating these structures, as a result, new invariant examples for the main classes in Hermitian and generalized Hermitian geometry are presented.

2. Almost Hermitian structures and metric f–structures

Let M be a smooth manifold, $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on M, d the exterior differentiation operator. An *almost Hermitian structure* on M (briefly, *AH–structure*) is a pair (g, J), where $g = \langle \cdot, \cdot \rangle$ is a (pseudo)Riemannian metric on M, J an almost complex structure such that $\langle JX, JY \rangle = \langle X, Y \rangle$ for any

$X, Y \in \mathfrak{X}(M)$. It follows immediately that the tensor field $\Omega(X,Y) = \langle X, JY \rangle$ is skew-symmetric, i.e. (M, Ω) is an almost symplectic manifold. Ω is usually called a *fundamental form* (the *Kähler form*) of an AH–structure (g, J).

Further, denote by ∇ the Levi-Civita connection of the metric g on M. We recall below some main classes of AH–structures together with their defining properties (see, for example, [12], [16]):

K	*Kähler structure*:	$\nabla J = 0$;
H	*Hermitian structure*:	$\nabla_X(J)Y - \nabla_{JX}(J)JY = 0$;
$\mathbf{G}_1$	AH–structure of class G_1, or G_1*–structure*:	$\nabla_X(J)X - \nabla_{JX}(J)JX = 0$;
QK	*quasi-Kähler structure*:	$\nabla_X(J)Y + \nabla_{JX}(J)JY = 0$;
AK	*almost Kähler structure*:	$d\,\Omega = 0$;
NK	*nearly Kähler structure,* or NK*–structure*:	$\nabla_X(J)X = 0$.

It is well known (see, for example, [12], [16]) that

$$\mathbf{K} \subset \mathbf{H} \subset \mathbf{G_1};\;\; \mathbf{K} \subset \mathbf{NK} \subset \mathbf{G_1};\;\; \mathbf{NK} = \mathbf{G_1} \cap \mathbf{QK};\;\; \mathbf{K} = \mathbf{H} \cap \mathbf{QK}.$$

It should be noted that all the classes above mentioned correspond to some subsets in the set $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ of all almost Hermitian structures on M (see [12]). In particular, the class $\mathbf{G_1}$ corresponds to the class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ in the notation of [12]. Other correspondences are the following:

$\mathbf{H} \longleftrightarrow \mathcal{W}_\mathbf{3} \oplus \mathcal{W}_\mathbf{4}$; $\mathbf{QK} \longleftrightarrow \mathcal{W}_\mathbf{1} \oplus \mathcal{W}_\mathbf{2}$; $\mathbf{NK} \longleftrightarrow \mathcal{W}_\mathbf{1}$; $\mathbf{AK} \longleftrightarrow \mathcal{W}_\mathbf{2}$; $\mathbf{K} \longleftrightarrow \{0\}$.
As usual, we will denote by N the Nijenhuis tensor of an almost complex structure J, that is,

$$N(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

for any $X, Y \in \mathfrak{X}(M)$. Then the condition $N = 0$ is equivalent to the integrability of J. Moreover, an almost Hermitian structure (g, J) belongs to the class **H** if and only if $N = 0$ (see, for example, [12]).

Furthermore, we will consider a more general concept of metric f–structures, which are a natural generalization of almost Hermitian structures.

An f*–structure* on a manifold M is known to be a field of endomorphisms f acting on its tangent bundle and satisfying the condition $f^3 + f = 0$ (see [23]). The number $r = dim\ Im\ f$ is constant at any point of M and called a *rank* of the f–structure.

Besides, the number $dim\, Ker\, f = dim\, M - r$ is usually said to be a *deficiency* of the f–structure and denoted by $def\, f$.

Recall that an f–structure on a (pseudo)Riemannian manifold $(M, g = \langle \cdot, \cdot \rangle)$ is called a *metric f–structure*, if $\langle fX, Y \rangle + \langle X, fY \rangle = 0,\ \ X, Y \in \mathfrak{X}(M)$ (see [16]). In this case the triple (M, g, f) is called a *metric f–manifold.* It is clear that the tensor field $\Omega(X, Y) = \langle X, fY \rangle$ is skew-symmetric, i.e. Ω is a 2–form on M. Ω is called a *fundamental form* of a metric f–structure [16]. It is easy to see that the particular cases $def\ f = 0$ and $def\ f = 1$ of metric f–structures lead to almost Hermitian structures and almost contact metric structures respectively.

Let M be a metric f–manifold. Then $\mathfrak{X}(M) = \mathcal{L} \oplus \mathcal{M}$, where $\mathcal{L} = Im\ f$ and $\mathcal{M} = Ker\ f$ are mutually orthogonal distributions, which are usually called the *first* and the *second fundamental distributions* of the f–structure respectively. Obviously, the endomorphisms $l = -f^2$ and $m = id + f^2$ are mutually complementary projections on the distributions $\mathcal{L}$ and $\mathcal{M}$ respectively. We note that in the case when the restriction of g to $\mathcal{L}$ is non-degenerate the restriction (F, g) of a metric f–structure to $\mathcal{L}$ is an almost Hermitian structure, i.e. $F^2 = -id,\ \langle FX, FY \rangle = \langle X, Y \rangle,\ X, Y \in \mathcal{L}$.

A fundamental role in the geometry of metric f–manifolds is played by the *composition tensor* T, which was explicitly evaluated in [16]:

$$T(X, Y) = \frac{1}{4} f(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y), \tag{2.1}$$

where ∇ is the Levi-Civita connection of a (pseudo)Riemannian manifold (M, g), $X, Y \in \mathfrak{X}(M)$. Using this tensor T, the algebraic structure of a so-called *adjoint Q-algebra* in $\mathfrak{X}(M)$ can be defined by the formula:

$$X * Y = T(X, Y).$$

It gives the opportunity to introduce some classes of metric f–structures in terms of natural properties of the adjoint Q–algebra [16]. We enumerate below the main classes of metric f–structures together with their defining properties [16], [6]:

Kf	*Kähler f–structure*:	$\nabla f = 0$;
Hf	*Hermitian f–structure*:	$T(X, Y) = 0$, i.e. $\mathfrak{X}(M)$ is an abelian Q–algebra;
$\mathbf{G_1 f}$	f–structure of class G_1, or *$G_1 f$–structure*:	$T(X, X) = 0$, i.e. $\mathfrak{X}(M)$ is an anticommutative Q–algebra;
QKf	*quasi-Kähler f–structure*:	$\nabla_X f + T_X f = 0$;

Kill f	*Killing f–structure*:	$\nabla_X(f)X = 0;$
NKf	*nearly Kähler f–structure,* or *NKf–structure*:	$\nabla_{fX}(f)fX = 0.$

The following relationships between the classes mentioned are evident:

$$\mathbf{Kf} = \mathbf{Hf} \cap \mathbf{QKf}; \quad \mathbf{Kf} \subset \mathbf{Hf} \subset \mathbf{G_1f}; \quad \mathbf{Kf} \subset \mathbf{Kill\ f} \subset \mathbf{NKf} \subset \mathbf{G_1f}.$$

It is important to note that in the special case $f = J$ we obtain the corresponding classes of almost Hermitian structures (see [12]). In particular, for $f = J$ the classes **Kill f** and **NKf** coincide with the well-known class **NK** of *nearly Kähler structures.*

Finally, we recall that the Nijenhuis tensor N_f of any f–structure is defined by

$$N_f(X, Y) = f^2[X, Y] + [fX, fY] - f[fX, Y] - f[X, fY]$$

for any $X, Y \in \mathfrak{X}(M)$ (see, for example, [23]). Besides, the characteristic condition for the integrability of f is $N_f = 0$ [23].

3. Canonical f–structures on homogeneous k–symmetric spaces

We briefly formulate some basic definitions and results related to regular Φ–spaces and canonical affinor structures on them. More detailed information can be found in [4], [7], [22], [17], [19], [20], [8] and others.

Let G be a connected Lie group, Φ its (analytic) automorphism, G^Φ the subgroup of all fixed points of Φ, and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition $G^\Phi_o \subset H \subset G^\Phi$. Then G/H is called a *homogeneous Φ–space.*

Homogeneous Φ–spaces include homogeneous symmetric spaces ($\Phi^2 = id$) and, more general, *homogeneous Φ–spaces of order k* ($\Phi^k = id$) or, in the other terminology, *homogeneous k–symmetric spaces* (see [17]).

For any homogeneous Φ–space G/H one can define the mapping

$$S_o = D\colon\ G/H \to G/H,\ xH \to \Phi(x)H.$$

It is known [19] that S_o is an analytic diffeomorphism of G/H. S_o is usually called a "symmetry" of G/H at the point $o = H$. It is evident that in view of homogeneity the "symmetry" S_p can be defined at any point $p \in G/H$.

Note that there exist homogeneous Φ–spaces that are not reductive. That is why so-called regular Φ–spaces first introduced by N.A.Stepanov [19] are of fundamental importance.

Let G/H be a homogeneous Φ–space, $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras for G and H, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A = \varphi - id$ and the Fitting decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition respectively. It is clear that $\mathfrak{h} = Ker\, A$, $\mathfrak{h} \subset \mathfrak{g}_0$. Recall that a homogeneous Φ-space G/H is called a *regular Φ–space* if $\mathfrak{h} = \mathfrak{g}_0$ [19]. Note that other equivalent defining conditions can be found in [4], [7].

We formulate two basic facts [19]:

Any homogeneous Φ–space of order k ($\Phi^k = id$) is a regular Φ–space.

Any regular Φ–space is reductive. More exactly, the Fitting decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \ \ \mathfrak{m} = A\mathfrak{g} \tag{3.1}$$

is a reductive one.

Decomposition (3.1) is called the *canonical reductive decomposition* corresponding to a regular Φ–space G/H, and $\mathfrak{m}$ is the *canonical reductive complement.* Besides, this decomposition is obviously φ–invariant. Denote by θ the restriction of φ to $\mathfrak{m}$. As usual, we identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$ at the point $o = H$. We note that θ commutes with any element of the linear isotropy group $Ad(H)$ (see [19]). It also should be noted (see [19]) that $(dS_o)_o = \theta$.

An *affinor structure* on a manifold is known to be a tensor field of type $(1,1)$ or, equivalently, a field of endomorphisms acting on its tangent bundle. Suppose F is an invariant affinor structure on a homogeneous manifold G/H. Then F is completely determined by its value F_o at the point o, where F_o is invariant with respect to $Ad(H)$. For simplicity, we will denote by the same manner both any invariant structure on G/H and its value at o throughout the rest of the paper.

Recall [4] that an invariant affinor structure F on a regular Φ–space G/H is called *canonical* if its value at the point $o = H$ is a polynomial in θ. It follows that any canonical structure is invariant, in addition, with respect to the "symmetries" $\{S_p\}$ of G/H.

Denote by $\mathcal{A}(\theta)$ the set of all canonical affinor structures on a regular Φ–space G/H. It is easy to see that $\mathcal{A}(\theta)$ is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. Evidently, the algebra $\mathcal{A}(\theta)$ for any symmetric Φ–space ($\Phi^2 = id$) is trivial, i.e. it is isomorphic to $\mathbb{R}$. As to arbitrary regular Φ-space $(G/H, \Phi)$, the algebraic structure of its commutative algebra $\mathcal{A}(\theta)$ was completely described (see [8]).

The most remarkable example of canonical structures is the canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on a homogeneous 3–symmetric space (see [20], [22], [10]). It turns out that it is not an exception. In other words, the algebra $\mathcal{A}(\theta)$ contains a rich collection of affinor structures of classical types. We will concentrate on the following affinor structures of classical types:

almost complex structures J ($J^2 = -1$);

almost product structures P ($P^2 = 1$);

f–structures ($f^3 + f = 0$) [23];

f–structures of hyperbolic type or, briefly, *h–structures* ($h^3 - h = 0$) [16].

Clearly, f–structures and h–structures are generalizations of structures J and P respectively.

All the canonical structures of classical type on regular Φ-spaces were completely described [4],[8]. In particular, for homogeneous k–symmetric spaces, precise computational formulae were indicated. The formulae for canonical f–structures are [4]:

$$f = \frac{2}{k}\sum_{m=1}^{u}\left(\sum_{j=1}^{u}\zeta_j \sin\frac{2\pi mj}{k}\right)\left(\theta^m - \theta^{k-m}\right), \tag{3.2}$$

where $u = \begin{cases} n, & \text{if } k = 2n+1 \\ n-1, & \text{if } k = 2n \end{cases}$, and $\zeta_j \in \{-1, 0, 1\}, j = 1, 2, \dots u$, moreover, not all of the numbers ζ_j are zero. The canonical f–structures f_j, determined by the sets $(\zeta_1, \dots \zeta_j, \dots \zeta_u)$, in which $\zeta_j = 1$ and the remaining components vanish are called *base* canonical f–structures.

The particular cases $k = 3, 4, 6$ of formula (3.2) are of special interest for our future consideration. In the case $k = 3$ we obtain the well-known canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ (see [20], [22], [10] and many other papers). As to homogeneous 4–symmetric spaces, they admit the only (up to sign) canonical f–structure $f = \frac{1}{2}(\theta - \theta^3)$. Note that the same structure was constructed (in a bit different form) in [18]. Finally, we indicate the formulae for all (up to sign) canonical f–structures on homogeneous 6–symmetric spaces:

$$f_1 = \frac{\sqrt{3}}{6}(\theta+\theta^2-\theta^4-\theta^5),\ f_2 = \frac{\sqrt{3}}{6}(\theta-\theta^2+\theta^4-\theta^5),\ f_3 = f_1+f_2,\ f_4 = f_1-f_2, \tag{3.3}$$

where the structures f_1 and f_2 are the base canonical f–structures.

It should be mentioned that canonical structures play an important role in Hermitian and generalized Hermitian geometry. Namely, certain classes of almost Hermitian structures are provided with the remarkable set of invariant examples by means of the canonical almost complex structure on homogeneous 3–symmetric spaces (see, e.g., [22], [10]). It turns out that there is also a wealth of invariant examples for the basic classes of metric f–structures. These invariant metric f–structures can be realized on homogeneous k–symmetric spaces with canonical f–structures (see, e.g., [18], [5], [6], [7], [8], [9]).

4. Canonical f–structures on the 6–dimensional generalized Heisenberg group

We briefly formulate some notions and results related to the 6–dimensional generalized Heisenberg group (N, g). As to details, we refer to [14], [15], [21].

Let V and Z be two real vector spaces of dimension n and m ($m \geq 1$) both equipped with an inner product which we shall denote for both spaces by the same symbol $\langle \cdot, \cdot \rangle$. Further, let $j : Z \to End(V)$ be a linear map such that

$$|j(a)x| = |x||a|, \ j(a)^2 = -|a|^2 I, \ x \in V, \ a \in Z.$$

Next we put $\mathfrak{n} := V \oplus Z$ together with the bracket defined by

$$[a + x, b + y] = [x, y] \in Z, \ \langle [x, y], a \rangle = \langle j(a)x, y \rangle,$$

where $a, b \in Z$ and $x, y \in V$. It is a 2–step nilpotent Lie algebra with center Z. The simply connected, connected Lie group N whose Lie algebra is $\mathfrak{n}$ is called a *generalized Heisenberg group.* Note that N has a left-invariant metric g induced by the following inner product on $\mathfrak{n}$:

$$\langle a + x, b + y \rangle = \langle a, b \rangle + \langle x, y \rangle, \ a, b \in Z, \ x, y \in V.$$

It is important to note that (N, g) is not a naturally reductive homogeneous space (see [15], [21], p.96). The Levi-Civita connection ∇ of the metric g was indicated in [14]:

$$\left\{ \begin{array}{c} \nabla_x y = \frac{1}{2}[x, y], \\ \nabla_a x = \nabla_x a = -\frac{1}{2} j(a)x \\ \nabla_a b = 0, \end{array} \right. , \tag{4.1}$$

where $a, b \in Z, \ x, y \in V$.

The 6–dimensional generalized Heisenberg group (N, g) is of especial interest (see [15], [21]). The brackets for the Lie algebra $\mathfrak{n} = L(x_1, x_2, x_3, x_4) \oplus L(a_1, a_2)$ were explicitly indicated in ([21], p.111):

$$(4.2)\qquad \begin{cases} [x_1, x_2] = a_1, \quad [x_1, x_3] = a_2, \\ [x_2, x_4] = -a_2, \quad [x_3, x_4] = a_1, \\ \text{all the other brackets being zero.} \end{cases}$$

In what follows, we use the interpretation of the 6–dimensional Lie algebra $\mathfrak{n}$ by means of quaternions (see [15], [21], p.104). More exactly, let $V = \mathbb{H}$ be the space of quaternions, and Z be a two-dimensional subspace of purely imaginary quaternions. Further, let $j : Z \to End\, V$ be the linear map defined by

$$j(a)x = a \cdot x, \quad a \in Z, \quad x \in V,$$

i.e. $j(a)x$ is the ordinary left-multiplication of x by a. We also note that $x_2 = j(a_1)x_1$, $x_3 = j(a_2)x_1$, $x_4 = j(a_1)j(a_2)x_1$ ([21], p.108).

It was shown in [21] that (N, g) is simultaneously 3– and 4–symmetric space. The canonical almost complex structure $J = \frac{1}{\sqrt{3}}(\theta - \theta^2)$ on the Riemannian 3–symmetric space (N, g), as it was stated in [21], is neither nearly Kähler nor almost Kähler. As to the canonical f–structure for the Riemannian 4–symmetric space (N, g), the following results were obtained:

Theorem 4.1. [5], [6] *The 6–dimensional generalized Heisenberg group (N, g) is a nearly Kähler and Hermitian f–manifold with respect to the canonical f–structure $f = \frac{1}{2}(\theta - \theta^3)$ of the Riemannian 4–symmetric space. This f–structure is not integrable on N.*

Following the method in [21], we construct an automorphism of order 6 for the Lie algebra $\mathfrak{n}$. Keeping the previous notations, we put [21]

$$U_1 = x_1 + ix_4, \;\; U_2 = x_2 + ix_3, \;\; U_3 = -a_1 + ia_2.$$

Further, define the linear map φ of $\mathfrak{n}$ by the formula:

$$\varphi(U_j) = e^{\frac{\pi i}{3}} U_j, \quad j = 1, 2, \quad \varphi(U_3) = -e^{\frac{\pi i}{3}} U_3.$$

It is not difficult to prove that φ is an isometric automorphism with the only fixed point of the Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$. Besides, $\varphi^6 = id$. If we consider the corresponding automorphism Φ of order 6 for the Lie group N, then (N, g) is a Riemannian homogeneous 6–symmetric space with respect to Φ. In our previous notations, $\theta = \varphi$.

Now, using formulae (3.3) for canonical f–structures, we can explicitely calculate the base f–structures in our case:

$$f_1 \ : \ (x_1, x_2, x_3, x_4, a_1, a_2) \ \to \ (-x_4, -x_3, x_2, x_1, 0, 0),$$

$$f_2 \ : \ (x_1, x_2, x_3, x_4, a_1, a_2) \ \to \ (0, 0, 0, 0, -a_2, a_1).$$

It easily follows that the other two canonical f–structures $f_3 = f_1 + f_2$ and $f_4 = f_1 - f_2$ are almost complex structures on N. Moreover, applying the general results from [7] we conclude the compatibility of all these structures with the metric g, i.e. f_1 and f_2 are metric f–structures as well as $J = f_3$ and $\tilde{J} = f_4$ are almost Hermitian structures.

Further, we notice that the structure $\tilde{J}$ is just the canonical almost complex structure for 3–symmetric space (N, g) mentioned above [21], [7]. Besides, the structure f_1 coincides with the canonical f–structure for the corresponding 4–symmetric space (see Theorem 4.1). It means that we should study the structures f_2 and J only.

Theorem 4.2. *Let (N, g) be the 6–dimensional generalized Heisenberg group considered as the Riemannian homogeneous 6–symmetric space. Then the canonical structure f_2 is a non-integrable nearly Kähler and Hermitian f–structure on the manifold N, but f_2 is not a Killing f–structure.*

Proof. The defining condition of the property **NKf** (see Section 2) for the structure f_2 is the following: $\nabla_{f_2X}(f_2)(f_2X) = 0$ for any $X \in \mathfrak{n}$. Put $X = x + a \in \mathfrak{n}$, where $x \in V, \ a \in Z$. Then, using formula (4.1), we obtain:

$$\nabla_{f_2X}(f_2)(f_2X) = \nabla_{f_2(a)}\,(-a) - f_2\ \nabla_{f_2(a)}\ f_2(a) = 0 - f_2(0) = 0.$$

It means that f_2 is an NKf–structure. However, it can be proved in the same manner that the condition $\nabla_X(f_2)X = 0$ is not satisfied, i.e. f_2 is not a Killing f–structure.

Further, we recall that the composition tensor T for any NKf–structure on a manifold $(M, \langle\cdot,\cdot\rangle, f)$ can be presented in the form ([8], p.121):

$$T(X, Y) = \frac{1}{2} f \nabla_{fX}(f) fY,$$

where $X, Y \in \mathfrak{X}(M)$. Put again $X = x + a, Y = y + b$, then we have:

$$\nabla_{f_2X}(f_2)(f_2Y) = \nabla_{f_2(a)}\,(-b) - f_2\ \nabla_{f_2(a)}\ f_2(b) = 0 - f_2(0) = 0.$$

It implies $T(X, Y) = 0$, i.e. f_2 is a Hermitian f–structure.

Finally, it remains to show that the Nijenhuis tensor N_f (see Section 2) of the f–structure f_2 is not trivial. Keeping the same notations and taking into account

(4.2), we can calculate the Nijenhuis tensor in our case:

$$N_{f_2}(X,Y) = f_2^2[x,y] = -[x,y] \neq 0.$$

As a result, f_2 is not integrable on (N,g). This completes the proof. □

Theorem 4.3. *The 6–dimensional generalized Heisenberg group (N,g) is a G_1–manifold with respect to the left-invariant canonical almost Hermitian structure $J = f_3 = f_1 + f_2$ of the Riemannian homogeneous 6–symmetric space (N,g,Φ). Besides, the structure J is neither nearly Kähler nor Hermitian structure on the manifold (N,g).*

Proof. We should verify the condition (see Section 2)

$$\nabla_X(J)X - \nabla_{JX}(J)JX = 0.$$

As above, put $X = x + a \in \mathfrak{n}$, where $x \in V$, $a \in Z$. Taking into account that $[x, Jx] = 0$ for any $x \in V$, we can obtain:

$$\nabla_X(J)X = -\frac{1}{2}(a \cdot Jx + Ja \cdot x - 2J(a \cdot x)).$$

Using the same method, we get:

$$\nabla_{JX}(J)JX = \frac{1}{2}(Ja \cdot x + a \cdot Jx + 2J(Ja \cdot Jx)).$$

Thus we have

$$\nabla_X(J)X - \nabla_{JX}(J)JX = -a \cdot Jx - Ja \cdot x + J(a \cdot x) - J(Ja \cdot Jx). \tag{4.3}$$

Directly calculating, one can get the following equality:

$$a \cdot Jx + Ja \cdot x = 0. \tag{4.4}$$

Now, substituting Jx for x in (4.4), we get

$$Ja \cdot Jx - a \cdot x = 0. \tag{4.5}$$

Finally, combining (4.3), (4.4), and (4.5), we obtain:

$$\nabla_X(J)X - \nabla_{JX}(J)JX = 0.$$

In addition, it is obvious from above that J is not a nearly Kähler structure. At last, the application of Theorems 4.1 and 4.2 yields that the structure $J = f_1 + f_2$ cannot be integrable. It means that J is not a Hermitian structure. This concludes the proof. □

Remark 4.1. It is interesting to note that the almost Hermitian structure $\tilde{J} = f_4 = f_1 - f_2$ is not a G_1–structure on (N, g).

Indeed, using the method of Theorem 4.3, we can get:

$$\nabla_X(\tilde{J})X - \nabla_{\tilde{J}X}(\tilde{J})\tilde{J}X = -2a \cdot \tilde{J}x + 2\tilde{J}(a \cdot x) \neq 0.$$

Remark 4.2. The class $\mathbf{G_1}$ of almost Hermitian structures is of especial interest in dimension 6. This class can be defined by the condition that the Nijenhuis tensor is totally skew-symmetric or, equivalently, there exists a linear connection ∇ preserving the almost Hermitian structure and with totally skew-symmetric torsion [3]. That is why G_1–structures have interesting applications in heterotic strings (see, for example, [13]). Specifically, in [13] the authors presented an example of a left-invariant proper (i.e. neither nearly Kähler nor Hermitian) G_1–structure on the 6–dimensional 2–step nilpotent Lie group G. Not going into details, we also note that the technique used in [13] is based on left-invariant 1–forms, and the group G constructed has a 3–dimensional center. Thus, the example as well as the method in [13] are completely different from our consideration in Theorem 4.3. It is very interesting to the author if there were in the literature other explicitly described examples of left-invariant proper (i.e. neither nearly Kähler nor Hermitian) G_1–structures on Lie groups, in particular, in the 6–dimensional case.

Remark 4.3. The 6–dimensional generalized Heisenberg group (N, g) corresponds to the real Lie group underlying the 3–dimensional complex Heisenberg group G_H. Further, the Iwasawa manifold $M = G_H/\Gamma$ is a compact nilmanifold , where Γ is the discrete subgroup generated by means of Gaussian integers (see, for example, [1]). Almost Hermitian structures on the Iwasawa manifold were intensively studied by many authors. In particular, in [1], [2] the authors considered the set $\mathcal{Z}$ of all invariant almost complex structures on M compatible with the standard metric g and the orientation of M, which is determined by the natural complex structure J_0 on the Iwasawa manifold. Specifically, in the notations of [1], [2], they proved that $\mathcal{Z}_1 = \emptyset$ (there are no nearly Kähler structures in $\mathcal{Z}$) and $\mathcal{Z}_{34} = \mathcal{Z}_{134}$ (the classes $\mathbf{G_1}$ and $\mathbf{H}$ coincide in $\mathcal{Z}$, i.e. any G_1–structure is integrable). It seems to be interesting to study interrelation between invariant metric f–structures (in particular, almost Hermitian structures) on the 6–dimensional generalized Heisenberg group and on the Iwasawa manifold.

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References

[1] E. Abbena, S. Garbiero, S. Salamon, *Hermitian geometry on the Iwasawa manifold*, Boll. Un. Mat. Ital. (7) **11-B** (1997), (2), 231–249.

[2] E. Abbena, S. Garbiero, S. Salamon, *Almost Hermitian geometry on six-dimensional nilmanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **30** (2001), (1), 147–170.

[3] B. Alexandrov, Th. Friedrich., N. Schoemann, *Almost Hermitian* 6*–manifolds revisited*, Journal of Geometry and Physics **53** (2005), (1), 1–30.

[4] V. V. Balashchenko and N. A. Stepanov, *Canonical affinor structures of classical type on regular* Φ*–spaces*, Matematicheskii Sbornik **186** (1995), (11), 3–34 (in Russian); English translation: Sbornik: Mathematics **186** (1995), (11), 1551–1580.

[5] V. V. Balashchenko, *Homogeneous Hermitian* f*–manifolds*, Uspekhi Mat. Nauk **56** (2001), (3), 159–160 (in Russian); English translation: Russian Math. Surveys **56** (2001), (3), 575–577.

[6] V. V. Balashchenko, *Invariant nearly Kähler* f*–structures on homogeneous spaces*, Contemporary Mathematics **288** (2001), 263–267.

[7] V. V. Balashchenko, *Invariant structures generated by Lie group automorphisms on homogeneous spaces*, Proceedings of the Workshop "Contemporary Geometry and Related Topics" (Belgrade, Yugoslavia, 15-21 May, 2002). Editors: N.Bokan, M.Djoric, A.T.Fomenko, Z.Rakic, J.Wess. World Scientific, 2004, 1–32.

[8] V.V. Balashchenko, Yu. G. Nikonorov, E. D. Rodionov, V. V. Slavsky, *Homogeneous spaces: theory and applications: monograph*, Polygrafist, Hanty-Mansijsk, 2008 (in Russian), $http://window.edu.ru/window\_catalog/files/r66129/160.pdf$.

[9] V. V. Balashchenko and A. S. Samsonov, *Nearly Kähler and Hermitian* f*–structures on homogeneous* k*–symmetric spaces*, Doklady Akademii Nauk **432** (2010), (3), 295–298 (in Russian); English translation: Doklady Mathematics **81** (2010), (3), 386–389.

[10] A. Gray, *Riemannian manifolds with geodesic symmetries of order* 3, J. Diff. Geom. **7** (1972), (3-4), 343–369.

[11] A. Gray, *Homogeneous almost Hermitian manifolds*, Proceedings of the Conference on Differential Geometry on Homogeneous Spaces, Turin, Italy, 1983; Rendiconti del Seminario Matematico Universita e Politecnico di Torino, (Special Issue) (1983), 17–58.

[12] A. Gray and L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura ed Appl. **123** (1980), (4), 35–58.

[13] P. Ivanov and S. Ivanov, $SU(3)$*–instantons and* $G_2, Spin(7)$*–heterotic string solitons*, Comm. Math. Phys. **259** (2005), (1), 79–102.

[14] A. Kaplan, *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata **11** (1981), 127–136.

[15] A. Kaplan, *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.

[16] V. F. Kirichenko, *Methods of generalized Hermitian geometry in the theory of almost contact manifolds*, Itogi Nauki i Tekhniki: Probl. Geom. **18** (1986), VINITI, Moscow, 25–71 (in Russian); English translation: J. Soviet Math. **42** (1988), (5), 1885–1919.

[17] O. Kowalski, *Generalized symmetric spaces*, LN in Math., **805**, Berlin, Heidelberg, New York: Springer-Verlag, 1980.

[18] A. J. Ledger, *Naturally reductive locally* 4*–symmetric spaces*, Israel J. Math. **63** (1988), (2), 223–236.

[19] N. A. Stepanov, *Basic facts of the theory of* φ*–spaces*, Izv. Vyssh. Uchebn. Zaved. Mat. **3** (1967), 88–95 (in Russian); English translation: Soviet Math. (Iz. VUZ) **11** (1967), (3).

[20] N. A. Stepanov, *Homogeneous* 3*–cyclic spaces*, Izv. Vyssh. Uchebn. Zaved. Mat. **12** (1967), 65–74 (in Russian); English translation: Soviet Math. (Iz. VUZ) **11** (1967), (12).

[21] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, London Math. Soc., LN Ser., **83**, 1983.

[22] J. A. Wolf and A. Gray, *Homogeneous spaces defined by Lie group automorphisms*, J. Diff. Geom. **2** (1968), (1-2), 77–159.

[23] K. Yano and M. Kon, *CR Submanifolds of Kaehlerian and Sasakian Manifolds*, Birkhäuser, Boston, Inc., 1983.

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