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HIGHER-DIMENSIONAL CENTRAL PROJECTION INTO 2-PLANE WITH VISIBILITY AND APPLICATIONS

J. KATONA [1], E. MOLNÁR [2], I. PROK [3], AND J. SZIRMAI [4]

Abstract. Applying d-dimensional projective spherical geometry $\mathcal{PS}^d(\mathbf{R}, \mathbf{V}^{d+1}, \boldsymbol{V}_{d+1})$, represented by the standard real $(d+1)$-vector space and its dual up to positive real factors as $\sim$ equivalence, the Grassmann algebra of $\mathbf{V}^{d+1}$ and of $\boldsymbol{V}_{d+1}$, respectively, represent the subspace structure of $\mathcal{PS}^d$ and of $\mathcal{P}^d$. Then the central projection from a $(d-3)$-centre to a 2-screen can be discussed in a straightforward way, but interesting visibility problems occur, first in the case of $d = 4$ as a nice attractive application. So regular 4-solids can be visualized in the Euclidean space $\mathbf{E}^4$ and non-Euclidean geometries, e.g. spherical $\mathbf{S}^4$ and hyperbolic $\mathbf{H}^4$ geometry. In a short report geodesics and geodesic spheres will also be illustrated in $\mathbf{H}^2 \times \mathbf{R}$ and $\widetilde{\mathbf{SL}_2\mathbf{R}}$ spaces by projective metric geometry.

1. Strategy

In the Vorau Conference on Geometry 2007 the first two authors presented the problem "Visibility of the higher-dimensional central projection onto the projective sphere" appeared later in *Acta Mathematica Hungarica* [5]. In that paper we gave a general procedure – implemented by the first author to the central projection of the 4-cube directly (without intermediate 3-projection) into the 2-plane of the computer screen – which projects the edge framework of a d-polytope onto a p-plane from a complementary s-centre-figure ($p + s + 1 = d$, e.g. $p = 2$, $s = 1$, $d = 4$ now). All these were embedded into the machinery of Grassmann (or Clifford) algebras of $d+1$-vector- and form- spaces, describing the projective metric d-spheres, initiated by

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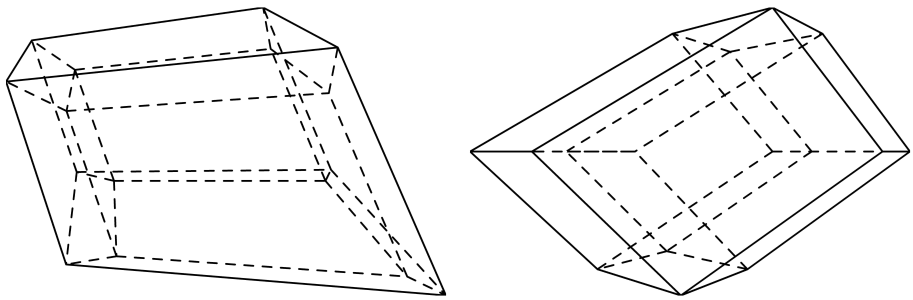


FIGURE 1. The dashed edges are not visible. Two square 2-faces cover the others in the first picture. 8 3-cubes form the 4-cube (8-cell). In the second picture 5 square 2-faces are visible. We see also 2 squares whose projections each degenerates into a line.

the second author. Thus, Euclidean and other (e.g. hyperbolic, spherical and other projective metric Thurston) geometries can also uniformly be discussed [9].

Now we specialize that procedure to the most important orthogonal (or parallel) projections of the regular 4-polytopes elaborated by the third author in his homepage [12] without any visibility, but 4-polytopes nicely move in the screen. Our initiative with visibility makes these demonstrations more attractive, and this seems to be new and timely procedure, not finished yet.

The fourth author and his students extended the visualization also to the other Thurston geometries, now $\mathbf{H}^3$, $\mathbf{H}^2\times\mathbf{R}$, and $\widetilde{\mathbf{SL_2R}}$ will be illustrated here.

In Figure 1, as motivation, the four-dimensional cube, with Schläfli symbol $(4, 3, 3)$ is pictured in central projection on the two-dimensional computer screen.

In Figure 2 the 2-dimensional projective sphere is embedded into the affine space $\mathcal{A}^3(O, \mathbf{V}, \boldsymbol{V})$. This scene can be thought also in the higher-dimensional situation.

2. A UNIFIED VECTOR CALCULUS

We can describe classical planes uniformly, when we embed these planes into the projective sphere. This method suits for discussing spherical, hyperbolic, Euclidean, Minkowskian and Galilean planes. Projective and affine planes will be special cases, too [9].

Let $\mathbf{V}^3 = \mathbf{V}$ be a vector space over the real numbers $\mathbf{R}$, and $\boldsymbol{V}_3 =: \boldsymbol{V}$ is its dual space or space of its linear forms. Let $\mathbf{a}_i$ be a basis in $\mathbf{V}$. Then $\boldsymbol{b}^j$ is its dual basis in $\boldsymbol{V}$, iff $\mathbf{a}_i\boldsymbol{b}^j = \delta_i^j$ (the Kronecker symbol). We consequently denote by

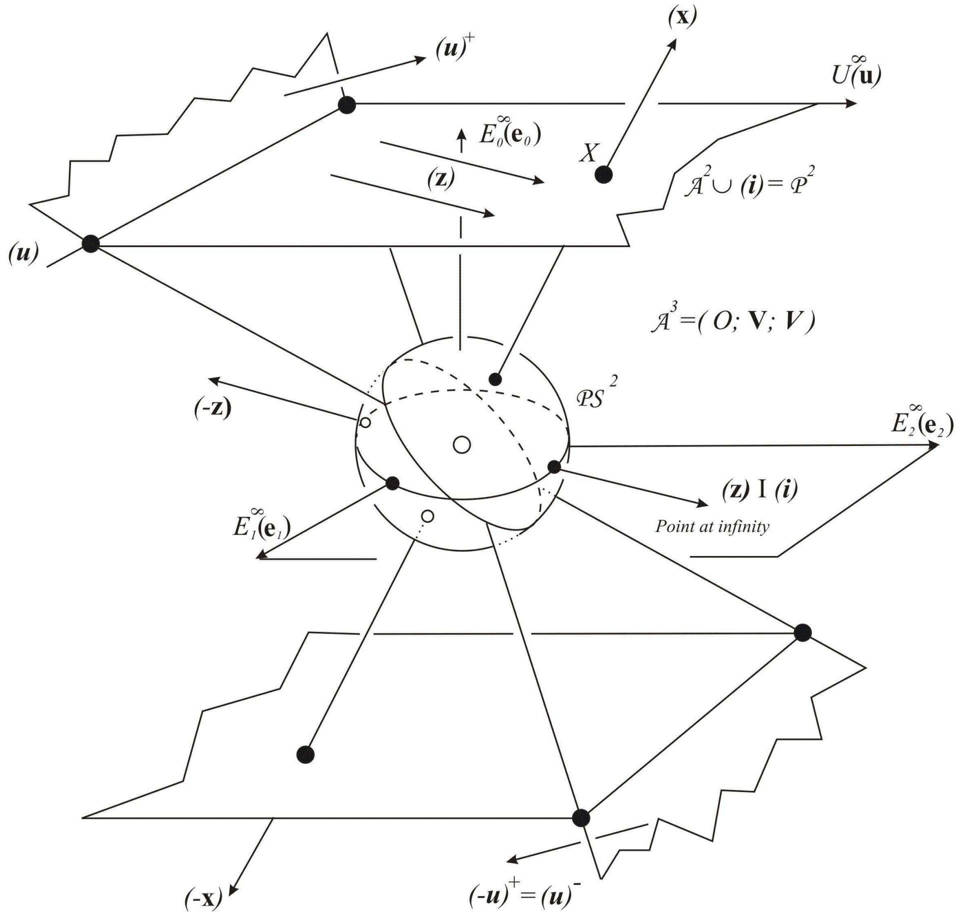


FIGURE 2. The projective sphere $\mathcal{PS}^2$, the double affine plane $\mathcal{A}^2$ and the projective plane $\mathcal{P}^2$ can also be visualized by vectors of $\mathbf{V}^3$ and of forms of $\boldsymbol{V}_3$ as follows.

$$\mathbf{x} = x^i\mathbf{a}_i = (x^0\ x^1\ x^2)\begin{pmatrix}\mathbf{a}_0\\ \mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix} \in \mathbf{V} \text{ and } \boldsymbol{u} = \boldsymbol{b}^j u_j = (\boldsymbol{b}^0\ \boldsymbol{b}^1\ \boldsymbol{b}^2)\begin{pmatrix}u_0\\ u_1\\ u_2\end{pmatrix} \in \boldsymbol{V}$$

the corresponding bases and coordinates of vectors and forms, respectively, and apply Einstein sum convention for the same upper and lower indices from 0 to 2.

Form $\boldsymbol{u} \in \boldsymbol{V}$ takes the value $\mathbf{x}\boldsymbol{u} = x^i u_i \in \mathbf{R}$ on $\mathbf{x} \in \mathbf{V}$. The vector class $\mathbf{x} \sim c\mathbf{x} \Rightarrow (\mathbf{x})$ defines a point $X = (\mathbf{x})$ in the projective sphere $\mathcal{PS}^2$ with $c > 0$ and $(\mathbf{x}) = (-\mathbf{x})$ in projective plane $\mathcal{P}^2$ with $c \in \mathbf{R} \setminus \{0\}$. In dual terms: $\boldsymbol{u} \sim \boldsymbol{u} \cdot \frac{1}{c} \Rightarrow (\boldsymbol{u})$ defines a (directed) line $u = (\boldsymbol{u})$ in $\mathcal{PS}^2$ iff $\frac{1}{c} > 0$; a line $u = (\boldsymbol{u}) = (-\boldsymbol{u})$ iff $\frac{1}{c} \in \mathbf{R} \setminus \{0\}$ for $\mathcal{P}^2$. The incidence $(\mathbf{x}) \in (\boldsymbol{u})$ means $\mathbf{x}\boldsymbol{u} = 0$. Figure 2 shows, how an affine plane A^2 is

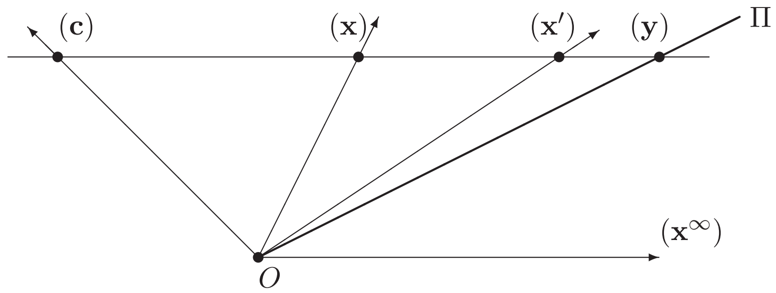


FIGURE 3. Two points with the same image by vector interpretation

embedded into an affine space $A^3(O;\mathbf{V},\boldsymbol{V})$, into the projective plane $\mathcal{P}^2 = A^2 \cup (\boldsymbol{i})$, furthermore, into the projective sphere $\mathcal{PS}^2$ that can be considered as a "double affine plane" extended by a "double ideal line" $(\boldsymbol{i})$ at infinity.

Let the main difference to the usual discussion be emphasized: in $\mathcal{PS}^2$ an affine line has two ideal points at infinity, one of them is distinguished, assigned by the viewing direction of the observer. Every point of the affine line is doubled in order to form a circle (see also Figure 2 and Figure 3). As our Figure 4 will indicate in the 4-space, visualized in the usual 3-space and in the figure plane (Figure 4). The observer "stands" in the vanishing hyperplane, looking ahead from $C_3(\mathbf{c}_3)$ in directions pointing to the positive halfspace where the target polytope and then behind (say for simplicity, without loss of generality) the picture plane are placed. We can follow these analogies for the d-dimensional space $\mathcal{PS}^d$ as well.

3. Geometric description

The 1-dimensional case $\mathcal{PS}^1$ is illustrated in the vector plane $\mathbf{V}^2$ (Figure 3). The centrum, i.e. view-point is described by the vector class $(\mathbf{c}) = \{k\mathbf{c} : 0 < k \in \mathbf{R}\}$ with fixed $\mathbf{c} \in \mathbf{V} \setminus \{0\}$. In the picture plane Π the point $(\mathbf{p}) = (\mathbf{y})$ is the projection of an arbitrary point $(\mathbf{x})$. This covers another point $(\mathbf{x}')$, in the usual description above, $\mathbf{x} \sim \gamma\mathbf{c} + \mathbf{y}$, $\mathbf{x}' \sim \gamma'\mathbf{c} + \mathbf{y}$ iff the inequalities $\gamma > \gamma' > 0$ hold. The observer $(\mathbf{c})$ looks in the viewing direction $(\mathbf{x})^\infty$ as one ideal point of $\mathcal{PS}^1$. The scene in the four-dimensional space is symbolically pictured in Figure 4.

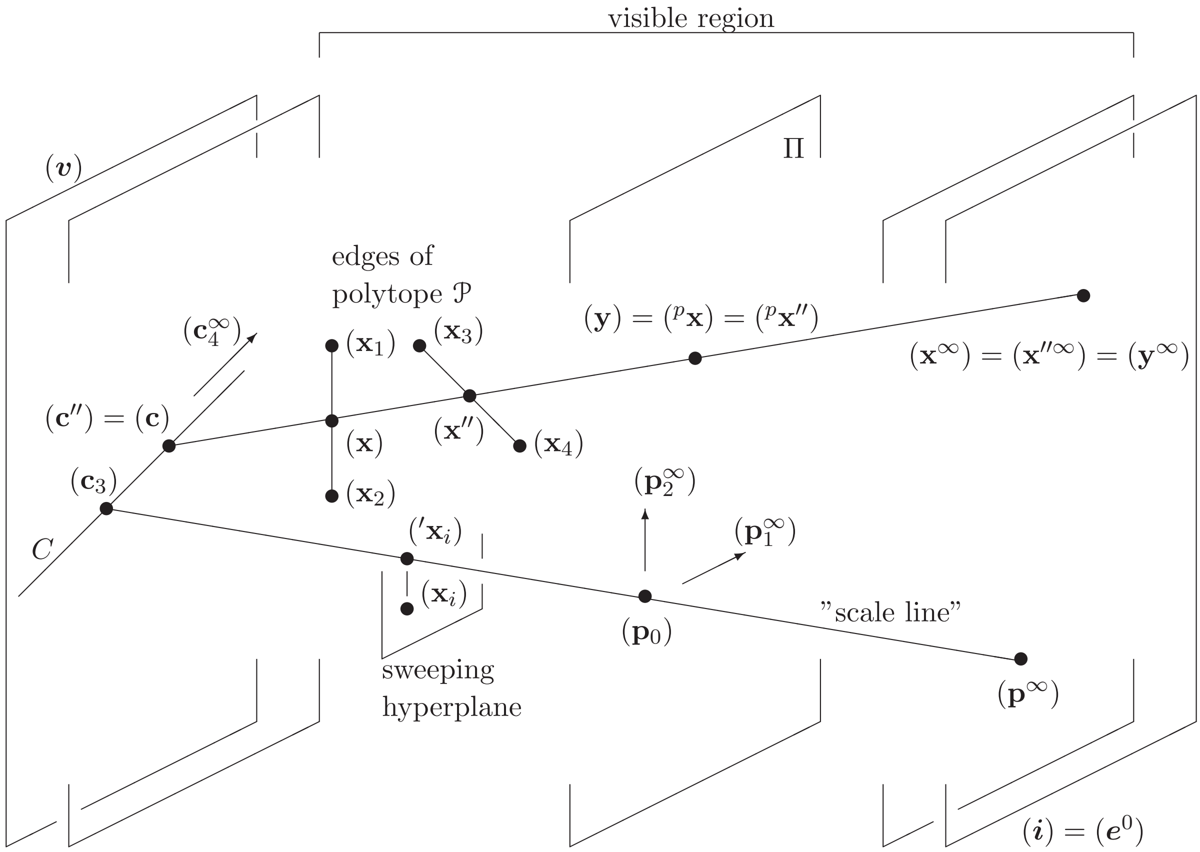


FIGURE 4. Ordering vertices to vanishing hyperplane $(\boldsymbol{v})$

An affin-projective coordinate simplex represents the camera by

$$\begin{pmatrix} \mathbf{p}_0 \\ \vdots \\ \mathbf{p}_p^\infty \\ \mathbf{c}_{p+1} \\ \vdots \\ \mathbf{c}_d^\infty \end{pmatrix} \sim \begin{pmatrix} 1 & \dots & p_0^p & p_0^{p+1} & \dots & p_0^d \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & p_p^p & p_p^{p+1} & \dots & p_p^d \\ c_{p+1}^0 & \dots & c_{p+1}^p & c_{p+1}^{p+1} & \dots & c_{p+1}^d \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & c_d^p & c_d^{p+1} & \dots & c_d^d \end{pmatrix} \begin{pmatrix} \mathbf{e}_0 \\ \vdots \\ \mathbf{e}_p^\infty \\ \mathbf{e}_{p+1}^\infty \\ \vdots \\ \mathbf{e}_d^\infty \end{pmatrix} \sim:$$

$$\sim: (\mathrm{Cam}) \begin{pmatrix} \mathbf{e}_0 \\ \vdots \\ \mathbf{e}_d^\infty \end{pmatrix}.$$

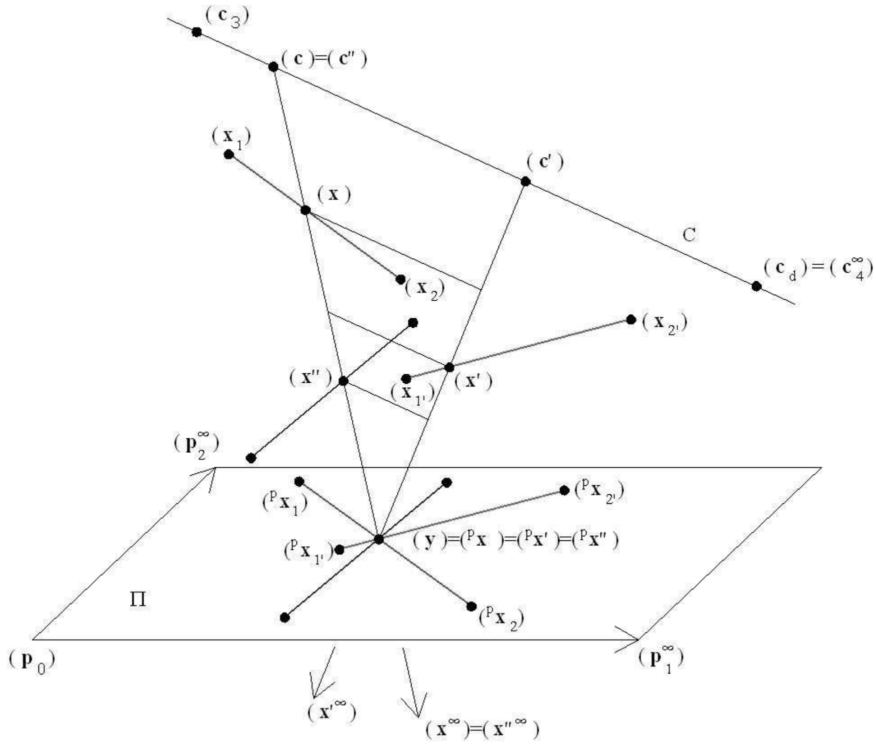


FIGURE 5. Projection of segments to extended local visibility

Here any point $X(\mathbf{x})$ in the visible region can be expressed as

$$\mathbf{x} \sim (1, x^1, \dots, x^p, x^{p+1}, \dots, x^d) \begin{pmatrix} \mathbf{e}_0 \\ \vdots \\ \mathbf{e}_d^\infty \end{pmatrix} \sim$$

$$\sim (y^0, y^1, \dots, y^p, c^{p+1}, \dots, c^d)(\mathrm{Cam}) \begin{pmatrix} \underline{\mathbf{e}}_0 \\ \vdots \\ \underline{\mathbf{e}}_d^\infty \end{pmatrix}, \quad \text{so that}$$

$$(1, x^1, \dots, x^p, x^{p+1}, \dots, x^d)(\mathrm{Cam})^{-1} \sim (y^0, y^1, \dots, y^p, c^{p+1}, \dots, c^d) \sim$$

$$(1, \frac{y^1}{y^0}, \dots, \frac{y^p}{y^0}, \frac{c^{p+1}}{y^0}, \dots, \frac{c^d}{y^0}).$$

Relative visibility of $X(\mathbf{x})$ to $X'(\mathbf{x}')$ with $(')$ coordinates can be decided by Figures 3-5 and by an ordering prescription:

a) *the images* $({}^p\mathbf{x}) = (\mathbf{y})$ *and* $({}^p\mathbf{x}') = (\mathbf{y}')$ *are different (both are visible);*

b) *if the images are the same, i.e.* $\mathbf{y} \sim \mathbf{y}'$, *namely* $\frac{y^1}{y^0} = \frac{y^{1'}}{y^{0'}}, \dots, \frac{y^p}{y^0} = \frac{y^{p'}}{y^{0'}}$, *then* $\frac{c^{p+1}}{y^0} > \frac{c^{(p+1)'}}{y^{0'}}$;

c) *if the above equalities hold, then* $\frac{c^d}{y^0} < \frac{c^{d'}}{y^{0'}}$ *(the reverse inequality holds for* $d = 4 = d'$*).*

Then $X(\mathbf{x})$ is nearer to the centre figure C than $X'(\mathbf{x}')$. We see here the critical points of our algorithm:

0, Premliminary triangulation of the polytope which will be projected;
1, Solution of too many linear equation systems (by Gauss-Seidel elimination);
2, Ordering the points to the centre figure C and picture plane Π (camera) by coordinates.

4. On global visibility. Triangulation in the projection procedure. New initiative, illustrated in $4 \to 2$ projection

For global visibility we compute and compare edges and 2-faces of a polyhedron (polytope) $\mathcal{P}$ by relative visibility.

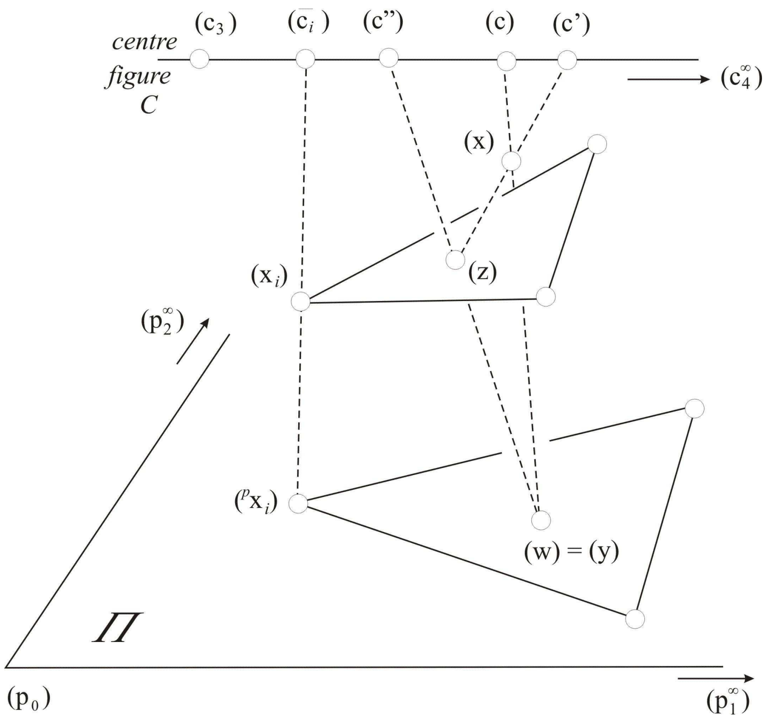


Figure 6

Namely, project an edge by its vertices, e.g. $(\mathbf{x})$ in Figure 6 into any 2-face $f = (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)$ of $\mathcal{P}$, as above in the former Section 3, in the coordinate simplex determined by $f = (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)$ and the centre figure $C = (\mathbf{c}_3 \wedge \mathbf{c}_4^\infty)$. Then project it further into the picture plane $\Pi(\mathbf{p}_0 \wedge \mathbf{p}_1^\infty \wedge \mathbf{p}_2^\infty)$. We use, only indicated here, a Grassmann algebra machinery with wedge product $\wedge$. Triangulations are assumed also for the 2-surface of $\mathcal{P}$ and for the picture plane Π. Let

$$\mathbf{x} \sim \mathbf{y} + \mathbf{c} \ \text{ with } (\mathbf{c}) \in C, \ (\mathbf{y}) \in \Pi$$
$$\mathbf{x} \sim \mathbf{z} + \mathbf{c}' \ \text{ with } (\mathbf{c}') \in C, \ (\mathbf{z}) \in f$$
$$\mathbf{z} \sim \mathbf{w} + \mathbf{c}\text{''} \ \text{ with } (\mathbf{c}\text{''}) \in C, \ (\mathbf{w}) = (\mathbf{y}) \in \Pi$$

be assumed, and analogously

$$\mathbf{x}_i \sim {}^p\mathbf{x}_i + \overline{\mathbf{c}}_i \ \text{ with } (\overline{\mathbf{c}}_i) \in C, \ ({}^p\mathbf{x}_i) \in \Pi \ (i = 0, \ 1, \ 2)$$

be assumed for the projection. Furthermore, let positive linear combination in

$$\mathbf{z} \sim z^0\mathbf{x}_0 + z^1\mathbf{x}_1 + z^2\mathbf{x}_2 \ \ (i.e. \ 0 < z^0, \ z^1, \ z^2)$$

and in

$$\mathbf{z} \sim {}^px^0\ {}^p\mathbf{x}_0 + {}^px^1\ {}^p\mathbf{x}_1 + {}^px^2\ {}^p\mathbf{x}_2 + \mathbf{c}\text{''} = \mathbf{w} + \mathbf{c}\text{''} \ \text{ with } (\mathbf{c}\text{''}) \in C, \ (\mathbf{w}) \in \Pi$$

be assumed, by vector independencies. Then $0 < z^0 = {}^px^0$, $z^1 = {}^px^1$, $z^2 = {}^px^2$ follow or the image of f degenerates, especially it is on the contour (i.e. $\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{c}_3 \wedge \mathbf{c}_4^\infty = 0$). We can conclude the following

Proposition 4.1. *The vertex $X(\mathbf{x})$ of $\mathcal{P}$ above is over the 2-face $f = (\mathbf{x}_0 \wedge \mathbf{x}_1 \wedge \mathbf{x}_2)$ of $\mathcal{P}$ related to camera $C \wedge \Pi$. iff above $0 < z^0, \ z^1, \ z^2$ hold, and for $\mathbf{c}' \sim c^{3'}\mathbf{c}_3 + c^{4'}\mathbf{c}_4^\infty$ in $\mathbf{x} \sim \mathbf{z} + \mathbf{c}'$ $c^{3'} > 0$ hold or if $c^{3'} = 0$ then $c^{4'} < 0$.*

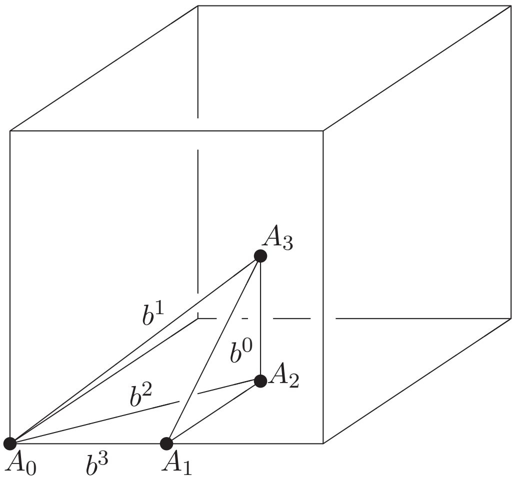
If $\mathcal{P}$ is a convex polytope then the visibility is more simple: It can start with the picture contour, as convex hull of the vertex images and with the top vertex of $\mathcal{P}$, by the ordering to centre figure C. Then visible edges, supported to the picture contour, can be determined.

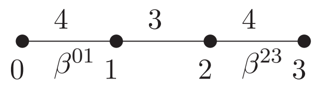
5. Coxeter-Schläfli diagram and matrix for 3-cube, 4-cube and regular d-polytopes

We illustrate the 3-cube in Figure 7 by its characteristic simplex: vertex A_0, edge centre A_1, face centre A_2, body centre A_3, and the 4 side faces, e.g. $b^0 = (A_1A_2A_3)$. That means e.g.

$$\cos\left[\pi - (b^1b^2)\right] = \cos\left[\pi - \frac{\pi}{3}\right] = -\frac{1}{2} = b^{12}$$

in the symmetric matrix (b^{ij}) $(i, j = 0, \ 1, \ 2, \ 3)$.



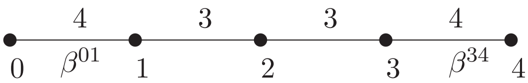


$$\begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 1 \end{pmatrix} = (b^{ij})$$

FIGURE 7. Cube in $\mathbb{E}^3$ and symbols for it

Analogous cases are collected in Tables 1-2 for the 4-cube and the regular d-polytope, respectively [4], [12].

TABLE 1

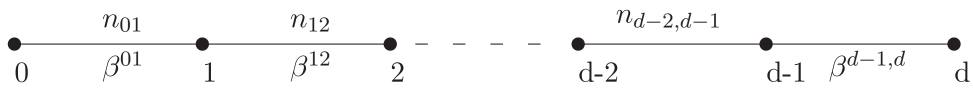


$$(b^{ij}) = \begin{pmatrix} 1 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 1 \end{pmatrix}$$

$$\begin{array}{c} B^{ij}_{44} : \\ \uparrow \\ (b^{-1}_{ij}) \\ \downarrow \\ B_{4j} \end{array} \begin{pmatrix} \frac{1}{2} & \frac{3\sqrt{2}}{8} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{8} & \frac{1}{8} \\ * & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & \frac{\sqrt{2}}{8} \\ * & * & \frac{1}{2} & \frac{1}{4} & \frac{\sqrt{2}}{8} \\ * & * & * & \frac{1}{4} & \frac{\sqrt{2}}{8} \\ \frac{1}{8} & \frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{8} & \frac{1}{8} \end{pmatrix}$$

TABLE 2

$\left\{n_{01}, n_{12}, \ldots, n_{d-2,d-1}; \beta^{d-1,d}\right\} \Longleftrightarrow$



$$b^{ij} = \begin{pmatrix} 1 & -\cos\beta^{01} & 0 & \dots & 0 & 0 \\ -\cos\beta^{01} & 1 & -\cos\beta^{12} & \dots & 0 & 0 \\ 0 & -\cos\beta^{12} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\cos\beta^{d-1,d} \\ 0 & 0 & 0 & \dots & -\cos\beta^{d-1,d} & 1 \end{pmatrix}$$

where $\beta^{ij} = \frac{\pi}{n_{ij}}$ for $i, j = 0, 1, \ldots, d;\ i \neq j,\ (i, j) \neq (d-1, d);\ 1 \leq n_{ij} \in \mathbb{N}$ natural numbers.

To a regular d-polytope $\mathcal{P}$ we introduce a characteristic simplex for $\mathcal{P}$ by the following general

Definition 5.1. We introduce an angle metric for our simplex S just by the starting bilinear form, considered as scalar product

$$\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = b^{ij} = \cos(\pi - \beta^{ij}), \quad i, j = 0, 1, \ldots, d.$$

Think of $\boldsymbol{b}^i$ as the inward "normal" unit vector to the facet b^i and so $\boldsymbol{b}^j$ to b^j as well.

We have a well known

Theorem 5.1. *The scalar product by b^{ij} above defines a spherical, hyperbolic or Euclidean angle metric of hyperplanes for the projective sphere $\mathcal{PS}^d$ by*

$$\cos\beta^{ij} = \frac{-b^{ij}}{\sqrt{b^{ii}b^{jj}}} \text{ or, in general,}$$

$$\cos\omega = \frac{-\langle \boldsymbol{u}; \boldsymbol{v} \rangle}{\sqrt{\langle \boldsymbol{u}; \boldsymbol{u} \rangle \langle \boldsymbol{v}; \boldsymbol{v} \rangle}} = \frac{-u_i b^{ij} v_j}{\sqrt{(u_r b^{rs} u_s)(v_r b^{rs} v_s)}}$$

for generalized dihedral angle ω of hyperplanes $(\boldsymbol{u})$ and $(\boldsymbol{v})$; according to the signature of b^{ij}:

$$\langle +, +, \ldots, +; + \rangle \text{ for spherical } d\text{-space } \mathbb{S}^d,$$

$$\langle +, +, \ldots, +; - \rangle \text{ for hyperbolic } d\text{-space } \mathbb{H}^d,$$

$$\langle +, +, \ldots, +; 0 \rangle \text{ for Euclidean } d\text{-space } \mathbb{E}^d.$$

By the inverse matrix of $(b^{ij}$ in case $\mathbf{S}^d$ and $\mathbf{H}^d$, i.e. by $(b^{ij})^{-1} = a_{ij}$, we can define the distance metric of simplex $A_0A_1 \dots A_d$, and in general, a coordinate presentation. $\mathbf{E}^d$ needs special discussion (in Table 1) by the minor subdeterminant matrix (B_{ij}) of (b^{ij}). For details see [4], [8].

6. Some pictures for non-Euclidean 3-spaces

We refer only to the possibilities of non-Euclidean 3-geometries, see our references for details.

1. Figures 8, 9 and 10 indicate a series of tilings on the base of group

$$\Gamma_p = \{r, z - r^2 = z^{2p} = rzrz^2rz^{-2}rz^{-1} = 1\}.$$

depending on the natural parameter $p \geq 3$. For $p = 3$ the tiling lies in $\mathbf{E}^3$ generated by the usual cube tiling. For $p \geq 4$ the tiling is hyperbolic in $\mathbf{H}^3$. The absolute figure is also shown by its shadow [13].

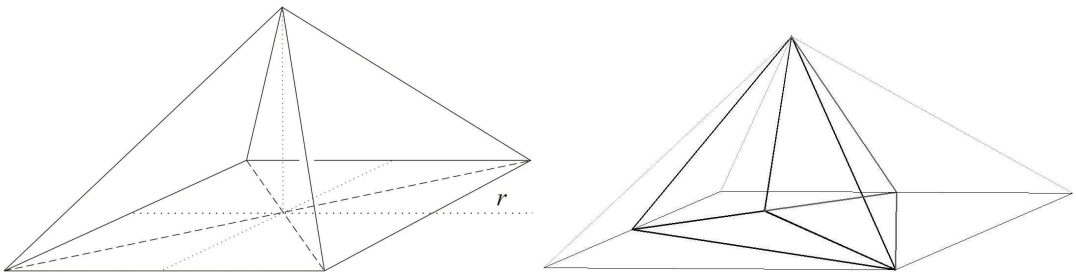


FIGURE 8. Euclidean case: $p = 3$ with a distinguished coordinate simplex

2. The points of $\mathbf{H}^2 \times \mathbf{R}$ space, in the projective space $\mathcal{P}^3$ forming an open cone solid, are the following:

$$\mathbf{H}^2 \times \mathbf{R} := \Big\{X(\mathbf{x} = x^i\mathbf{e}_i) \in \mathcal{P}^3 : -(x^1)^2 + (x^2)^2 + (x^3)^2 < 0 < x^0,\ x^1\Big\}.$$

On this base, the equation of geodesic lines in $\mathbf{H}^2 \times \mathbf{R}$ can be derived in usual Euclidean model coordinates $x = \frac{x^1}{x^0}$, $y = \frac{x^2}{x^0}$, $z = \frac{x^3}{x^0}$ as follows

$$x(\tau) = e^{\tau \sin v} \cosh{(\tau \cos v)},$$

$$y(\tau) = e^{\tau \sin v} \sinh{(\tau \cos v)} \cos u,$$

$$z(\tau) = e^{\tau \sin v} \sinh{(\tau \cos v)} \sin u,$$

$$-\pi < u \leq \pi, \quad -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}.$$

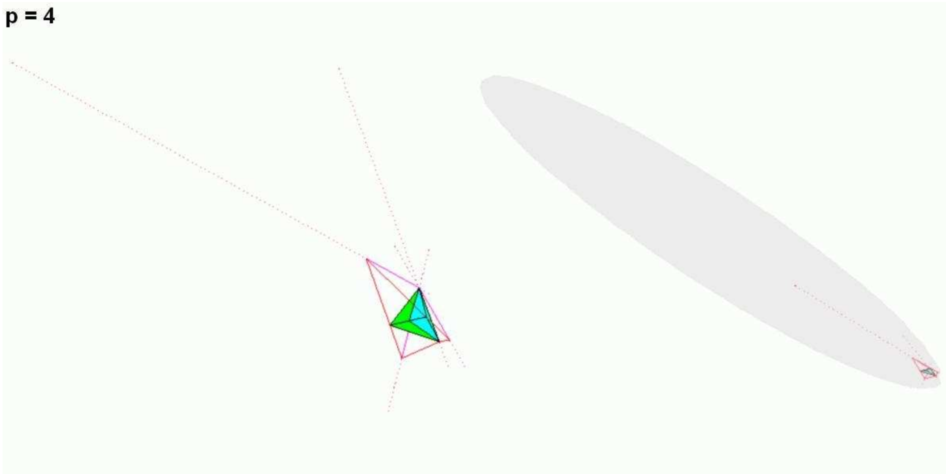


FIGURE 9. The hyperbolic case: $p = 4$. The absolute figure of $\mathbf{H}^3$ is indicated by its shadow

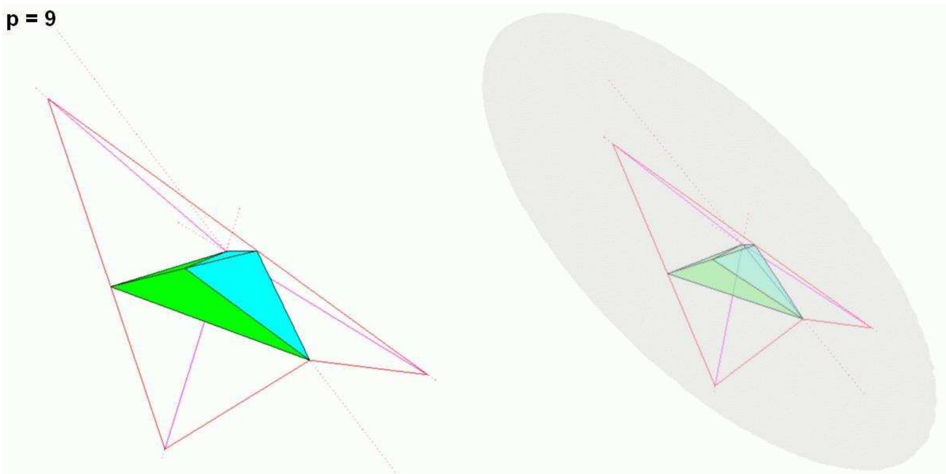


FIGURE 10. The former fundamental domain is shown in the hyperbolic case: $p = 9$ to the coodinate simplex. The absolute figure of $\mathbf{H}^3$ is indicated by its shadow

Here τ means the arc-length parameter, u and v are the geographic longitude and altitude, respectively, fixing the starting direction with unit velocity. If we fix $\tau = R$ in the above equations and vary u and v then we get the sphere of radius R with centre $(x_0,\ y_0,\ z_0) = (1,\ 0,\ 0)$. See Figure 11 where the zero level $\mathbf{H}_0^2$ is also indicated, as one part of the two-sheet hyperboloid. The half cone does not appear. Some special problems are discussed in [10], [11] and [14].

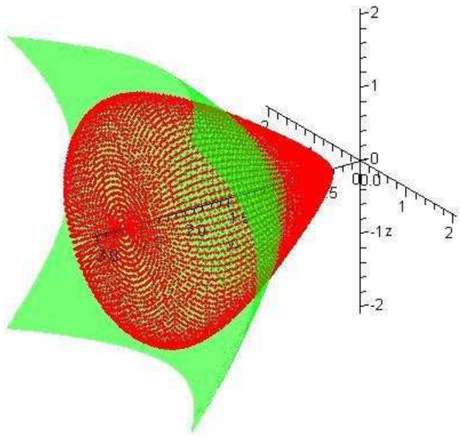


FIGURE 11. Geodesic ball in $\mathbf{H}^2\times\mathbf{R}$

3. Now we deal with $\widetilde{\mathbf{SL}_2\mathbf{R}}$ geometry [9], [10], [11]. In classical sense the 2×2 matrices, say now

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} \text{ with unit determinant } \; ad-bc=1$$

have 3 parameters for a 3-geometry. To have a more geometrical interpretation in the projective 3-sphere $\mathcal{PS}^3$, we introduce new coordinates (x^0,x^1,x^2,x^3), say by

$$a := x^0+x^3,\; b := x^1+x^2,\; c := -x^1+x^2,\; d := x^0-x^3,$$

with positive equivalence as projective freedom. Then

$$0 > bc-ad = -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3$$

describe the same 3-dimensional point set, namely the interior of the above unparted (one-sheet) hyperboloid (Figure 12). Indeed, for $x^0\neq 0$, $x=\frac{x^1}{x^0}$, $y=\frac{x^2}{x^0}$, $z=\frac{x^3}{x^0}$ we get the Euclidean model of $\widetilde{\mathbf{SL}_2\mathbf{R}}$

We introduce also a so-called hyperboloid parametrization as follows

$$X(x^0=\cosh r\cos(\phi),\; x^1=\cosh r\sin(\phi),$$
$$x^2=\sinh r\cos(\theta-\phi),\; x^3=\sinh r\sin(\theta-\phi),$$

$$(ds)^2=(dr)^2+(d\theta)^2\cosh^2(r)\sinh^2(r)+\Big[d\phi+(d\theta)\sinh^2(r)\Big].$$

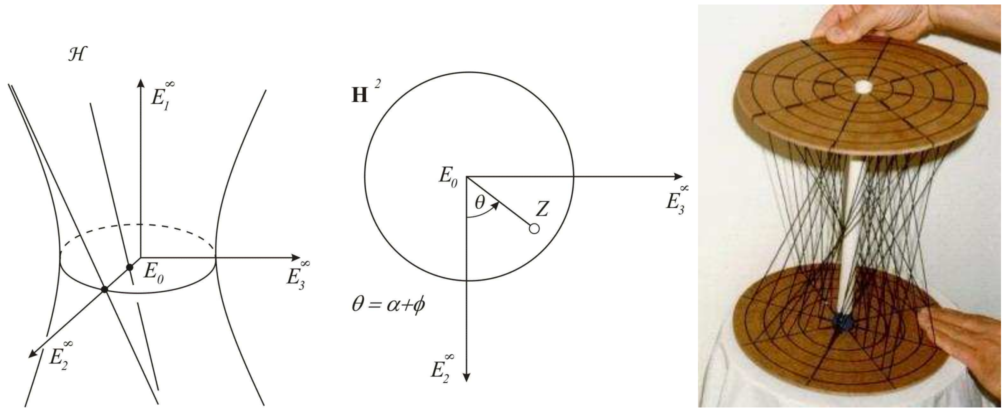


FIGURE 12. The unparted hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbf{R}}$ space of skew line fibres growing in points of the hyperbolic base plane. The gum-fibre model is due to Hans Havlicek and Rolf Riesinger, used also by Hellmuth Stachel with other respects

From this we get the differential equation of geodesics:

$$\ddot{r} = \sinh(2r)\dot{\theta}\dot{\phi} + \frac{1}{2}\Big[\sinh(4r) - \sinh(2r)\Big]\dot{\theta}\dot{\theta},$$
$$\ddot{\theta} = \frac{-2r}{\sinh(2r)}\Big[3(\cosh(2r) - 1)\dot{\theta} + 2\dot{\phi}\Big],$$
$$\ddot{\phi} = 2\dot{r}\tanh(r)\Big[2\sinh^2(r)\dot{\theta} + \dot{\phi}\Big],$$

with appropriate initial values for starting points and unit velocity (see [3], [11]).

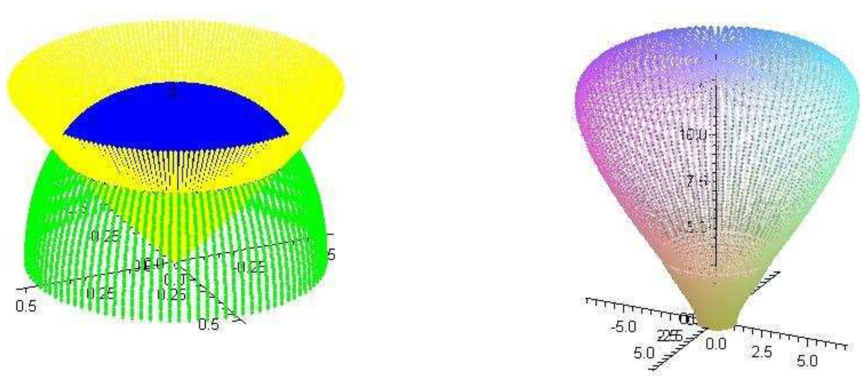


FIGURE 13. Geodesic half sphere of radius 0.5 with "cone" and geodesic half sphere of radius 1.5 in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ space

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[1] Ybl Miklós Faculty of Architecture,
Szent István University,
H-1146 Budapest, Thököly u. 74.
Hungary
E-mail address: katona.janos@ymmfk.szie.hu

[2,3,4] Department of Geometry,
Budapest University of Technology,
H-1111 Budapest, Egry József u. 1. H. II. 22.
Hungary
E-mail address: emolnar@math.bme.hu
E-mail address: prok@math.bme.hu
E-mail address: szirmai@math.bme.hu