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ON SCALAR AND TOTAL SCALAR CURVATURES OF RIEMANN-CARTAN MANIFOLDS

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Abstract. The concept of the Riemann–Cartan manifold was introduced by E. Cartan. The Riemann–Cartan manifold is a triple $(M, g, \bar{\nabla})$, where (M, g) is a Riemann n-dimensional $(n \geq 2)$ manifold with linear connection $\bar{\nabla}$ having nonzero torsion $\bar{S}$ such that $\bar{\nabla}g = 0$. In our paper, we have considered scalar and total scalar curvatures of the Riemann–Cartan manifold $(M, g, \bar{\nabla})$ and proved some formulas connecting these curvatures with scalar and total scalar curvatures of the Riemannian (M, g). In particular we have analyzed these formulas for the case of Weitzenbök manifolds. And in an inference we have proved some vanishing theorems.

1. Introduction

1.1. **A brief history of metrically-affine spaces.** The beginning of metrically-affine spaces (manifolds) theory was marked by E. Cartan in 1923–1925, who suggested using an asymmetric linear connection $\bar{\nabla}$ having the metric property $\bar{\nabla}g = 0$ (see [6], [7] and [8]). His theory was called Einstein–Cartan theory of gravity (ECT) (see [33]).

The notion of absolute parallelism or teleparallelism was introduced by Einstein in 1928 and 1930 when he tried to unify gravitation and electromagnetism. The new variant of a gravitational theory is formulated on the Weitzenböck space-time, characterized by the vanishing curvature tensor (absolute parallelism) $\bar{R} = 0$ and by the non-zero torsion tensor $\bar{S}$ (see [1] and [36]).

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More than thirty years after the event T. Kibble and D. Sciama have found a connection between the torsion $\bar{S}$ of the connection $\bar{\nabla}$ and the spin tensor S of matter (see [17] and [30]). Subsequently, other physical applications of ECT were found (see [24] and [27]).

The Einstein–Cartan theory was generalized by omitting the metric property of the linear connection $\bar{\nabla}$, i.e., the nonmetricity tensor $\mathcal{Q} = \bar{\nabla} g \neq 0$. The new theory was called the metrically-affine gauge theory of gravity (MAG) (see [15]).

1.2. **A geometrical aspect of brief history of metrically-affine spaces.** The idea of E. Cartan was reflected in the well-known books in differential geometry of the first half of the last century (see [10], [11], [28] and [29]). Now, there are hundreds works published in the frameworks of ECT and MAG, and moreover, the published results are of applied physical character (see [16] and [25]).

For a long time, among all forms metrically-affine space, only quarter-symmetric metric spaces and the semi-symmetric metric spaces were considered in differential geometry (see, for example, [22] and [37]).

The development of geometry of metrically-affine spaces “in the large” was stopped at the results of K. Yano, S. Bochner and S. Goldberg obtained in the middle of the last century. In their works, in the frameworks of RCT, they proved “vanishing theorems” for pseudo-Killing and pseudo-harmonic vector fields and tensors on compact Riemann-Cartan manifolds with positive-definite metric tensor g and the torsion tensor $\bar{S}$ such that $\operatorname{trace} \bar{S} = 0$ (see [3], [13] and [38]). Y. Kubo, N. Rani and N. Prakash have generalized their results by introducing in consideration compact Riemann–Cartan manifolds with boundary (see [20] and [26]). The following works are closely connected with this topic [21] and [35].

2. Riemann–Cartan manifolds

2.1. **Definition and trivial properties of the Riemann-Cartan manifold.** A *Riemann–Cartan manifold* is a triple $(M, g, \bar{\nabla})$, where (M, g) is a Riemannian n-dimensional ($n \geq 2$) manifold with linear connection $\bar{\nabla}$ having nonzero torsion $\bar{S}$ such that $\bar{\nabla} g = 0$ (see [33]), i.e. $\bar{\nabla}$ is a nonsymmetric metric connection on (M, g).

The deformation tensor T defined by $T = \bar{\nabla} - \nabla$, where ∇ is the Levi-Civita connection on (M, g) has the following properties (see [38]):

(i) T is uniquely defined,

(ii) $\bar{S}(X,Y) = 1/2\,(T(X,Y) - T(Y,X))$,
(iii) $T \in C^\infty TM \otimes \Lambda^2 M$ since $\bar{\nabla} g = 0 \Leftrightarrow g(T(X,Y),Z) = -g(T(X,Z),Y)$,
(iv) $g(T(Y,Z),X) = g(S(X,Y),Z) + g(S(X,Z),Y) + g(S(Y,Z),X)$,
(v) $\operatorname{trace} T = 2 \cdot \operatorname{trace} \bar{S}$

for any vector fields $X, Y, Z \in C^\infty TM$.

Let $\bar{R}$ be the curvature tensor of a nonsymmetric metric connection $\bar{\nabla}$. Then the covariant tensor $\bar{R}^b$ defined by the formula $\bar{R}^b(X,Y,Z,V) = g(\bar{R}(X,Y)V,Z)$ is a smooth section of the tensor bundle $\Lambda^2 M \otimes \Lambda^2 M$ (see [38]).

2.2. **Examples of Riemann-Cartan manifolds.** Firstly, we shall consider a Euclidian sphere $\mathring{\mathbf{S}}^2 = \{\mathbf{S}^2 \setminus \text{north pole}\}$ of radius R excluding the north pole and with the standard Riemannian metric $g_{11} = R^2\cos^2\varphi$, $g_{22} = R^2$, $g_{12} = g_{21} = 0$ where $x^1 = \vartheta$, $x^2 = \varphi$ denote the standard spherical coordinates of $\mathring{\mathbf{S}}^2$. Then $X_1 = \{(R\cos\varphi)^{-1}, 0\}$, $X_2 = \{0, R^{-1}\}$ are vectors of standard orthogonal basis of all vector fields on $\mathring{\mathbf{S}}^2$. There is a nonsymmetric metric connection $\bar{\nabla}$ with coefficients $\bar{\Gamma}^1_{21} = -\tan\varphi$ and other $\bar{\Gamma}^\alpha_{\beta\gamma} = 0$ such that $\bar{\nabla}_{X_\alpha} X_\beta = 0$ where $\alpha, \beta, \gamma = 1, 2$. For this connection $\bar{\nabla}$ the curvature tensor $\bar{R} = 0$ and the torsion tensor $\bar{S}$ has components $\bar{S}^1_{12} = \tan\varphi$, $\bar{S}^2_{12} = 0$ (see [7]). Therefore, $\mathring{\mathbf{S}}^2$ with g and $\bar{\nabla}$ is an example of a Riemann–Cartan manifold $(M, g, \bar{\nabla})$. In addition, if $\bar{R} = 0$, then we call the connection $\bar{\nabla}$ as *Weitzenbock* or *teleparallel connection* (see, for example, [1] and [36]).

Secondary, we shall consider a *homogeneous Riemannian manifold* (M, g) as the connected Riemannian manifold (M, g) whose isometry group is transitive. By the Ambrose–Singer theorem, a complete connected Riemannian manifold (M, g) is homogeneous if and only if a tensor field $T \in C^\infty TM \otimes \Lambda^2 M$ satisfies $\bar{\nabla} R = 0$ and $\bar{\nabla} T = 0$ for the connection $\bar{\nabla} = \nabla + T$. In this case, $\bar{\nabla} g = 0$ and, therefore, a homogeneous Riemannian manifold is an example of the Riemann–Cartan manifold $(M, g, \bar{\nabla})$ (see [34]).

Thirdly, we shall consider *almost Hermitian manifold* (see [19]) which is defined as the triple (M, g, J), where the pair (M, g) is a Riemannian $2m$-dimensional manifold with almost complex structure $J \in C^\infty(TM \otimes T^*M)$ compatible with the metric g, i.e. $J^2 = \mathrm{Id}_M$ and $g(J, J) = g$. In this case, $\bar{\nabla} g = 0$ for the connection $\bar{\nabla} = \nabla + \nabla J$, and, therefore, an almost Hermitian manifold (M, g, J), together with the connection $\bar{\nabla} = \nabla + \nabla J$, is an example of the Riemann–Cartan manifold $(M, g, \bar{\nabla})$.

The classification of almost Hermitian manifolds is well known (see [14]), it is based on the pointwise $U(m)$-irreducible decomposition of the tensor $\nabla\Omega$, where $\Omega(X,Y) = g(X,JY)$. *Almost semi-Kählerian manifold* (see [14]) is isolated by the condition $\operatorname{trace}\nabla J = 0$ and it is an example of Riemann–Cartan manifolds of class $\Omega_1 \oplus \Omega_3$. *Almost Kählerian manifold* (see [14]) is isolated by the condition $d\Omega = 0$ and it is an example of Riemann-Cartan manifolds of class $\Omega_2 \oplus \Omega_3$. *Nearly Kählerian manifold* (see [14]) is isolated by the condition $d\Omega = 3\nabla\Omega$ and it is example of Riemann–Cartan manifolds of class Ω_1.

2.3. **The first classification of Riemann–Cartan manifolds.** It is well-known that the torsion tensor $\bar{S} \in C^\infty \Lambda^2 M \otimes TM$. In turn, the following point-wise $O(q)$-irreducible decomposition holds (see [4])

$$\Lambda^2 M \otimes T^* M \cong \Omega_1(M) \otimes \Omega_2(M) \otimes \Omega_3(M).$$

Here, $q = g(x)$ for an arbitrary point $x \in M$. In this case, the orthogonal projections on the components of this decomposition are defined by the following relations (see [4]):

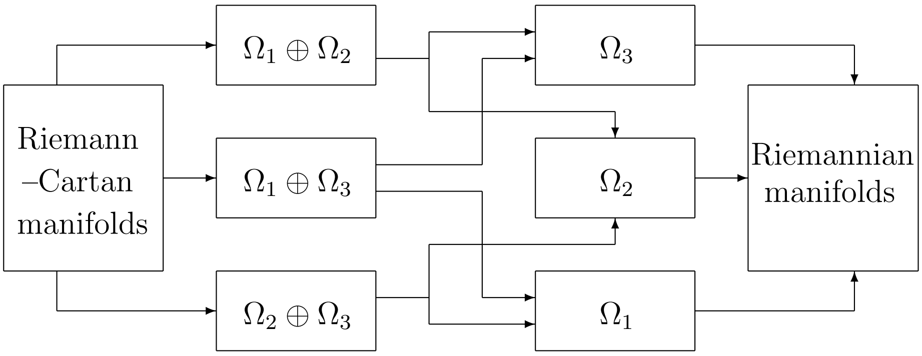
$$\begin{aligned}
&{}^{(1)}\bar{S}^b(X,Y,Z) = 1/3\left(\bar{S}^b(X,Y,Z) + \bar{S}^b(Y,Z,X) + \bar{S}^b(Z,X,Y)\right),\\
&{}^{(2)}\bar{S}^b(X,Y,Z) = g(X,Z)\theta(Y) - g(X,Y)\theta(Z),\\
&{}^{(3)}\bar{S}^b(X,Y,Z) = \left(\bar{S}^b(X,Y,Z) - {}^{(1)}\bar{S}^b(Y,Z,X) - {}^{(2)}\bar{S}^b(Z,X,Y)\right),
\end{aligned}$$

for any vector fields $X, Y, Z \in C^\infty TM$. In these identities, we have supposed that $\bar{S}^b(X,Y,Z) = g\left(\bar{S}(X,Y), Z\right)$ and $\theta := \frac{1}{n-1} \operatorname{trace} \bar{S}$.

We say that a Riemann–Cartan manifold $(M, g, \bar{\nabla})$ as well as its connection $\bar{\nabla}$ belong to the class Ω_α or $\Omega_\alpha \oplus \Omega_\beta$ for $\alpha, \beta = 1, 2, 3$ and $\alpha < \beta$ if the tensor field $\bar{S}^b$ is a section of corresponding tensor bundle $\Omega_\alpha(M)$ or $\Omega_\alpha(M) \oplus \Omega_\beta(M)$ (see also [5]).

Obviously, the Riemann–Cartan manifold $(M, g, \bar{\nabla})$ belongs to the class Ω_1 if and only if its torsion tensor satisfies the property $\bar{S}^b \in C^\infty \Lambda^3 M$. In particular this class includes spaces of semi-simple groups (see [11, 38]). Moreover the class Ω_2 of the Riemann–Cartan manifolds $(M, g, \bar{\nabla})$ consists of semi-symmetric Riemann–Cartan manifolds (see [2, 9, 22, 23, 31]).

All these classes of Riemann–Cartan manifolds are presented in the following diagram. Remark that any one can find physical interpretations of all classes of Riemann–Cartan manifolds in the paper [5].



2.4. **The second classification of Riemann–Cartan manifolds.** Pursuant to the Sec. 2.3 we conclude that deformation tensor $T^b \in C^\infty(TM \otimes \Lambda^2 M)$. In turn, the following pointwise $O(q)$-irreducible decomposition holds (see [4])

$$\Lambda^2 M \otimes T^* M \cong \Psi_1(M) \oplus \Psi_2(M) \oplus \Psi_3(M),$$

where the orthogonal projections on the components of this decomposition are defined by the following relations:

$$
\begin{aligned}
&{}^{(1)}T^b(X,Y,Z) = 1/3\Big(T^b(X,Y,Z) + T^b(Y,Z,X) + T^b(Z,X,Y)\Big),\\
&{}^{(2)}T^b(X,Y,Z) = g(X,Z)\omega(Y) - g(X,Y)\omega(Z),\\
&{}^{(3)}T^b(X,Y,Z) = \Big(T^b(X,Y,Z) - {}^{(1)}T^b(Y,Z,X) - {}^{(2)}T^b(Z,X,Y)\Big),
\end{aligned}
$$

for $T^b(X,Y,Z) = g\Big(\bar S(X,Y),Z\Big)$, $\omega = \dfrac{1}{n-1}\,\operatorname{trace} T$ and any vector fields $X, Y, Z \in C^\infty TM$.

We say (see [34]) that a Riemann–Cartan manifold $(M, g, \bar\nabla)$ as well as its connection $\bar\nabla$ belong to the class Ψ_α or $\Psi_\alpha \oplus \Psi_\beta$ for $\alpha, \beta = 1, 2, 3$ and $\alpha < \beta$ if the tensor field T^b is a section of corresponding tensor bundle $\Psi_\alpha(M)$ or $\Psi_\alpha(M) \oplus \Psi_\beta(M)$.

Obviously, the spaces $\Lambda^2 M \otimes T^* M$ and $T^* \otimes \Lambda^2 M$, as well as their irreducible components, are isomorphic. Therefore these two classifications are equivalent. Moreover, corresponding classes of Riemann–Cartan manifolds from these two classifications coincide.

2.5. **Green's theorem for the Riemann–Cartan manifolds.** Let (M, g) be a compact Riemannian manifold. We may also assume that (M, g) is orientable; if (M, g) is not orientable then we take an orientable twofold covering space of (M, g). The classical Green's theorem $\int(\operatorname{div} X)\, dV = 0$ has the form $\int(\operatorname{trace} \nabla X)\, dV = 0$ for an arbitrary smooth vector field X and the volume element dV of the Riemannian manifold (M, g) (see [38]).

Since the dependence $\bar{\nabla} = \nabla + T$ holds on a Riemann–Cartan manifold $(M, g, \bar{\nabla})$, it follows that trace $\bar{\nabla} X =$ trace $\nabla X + 2(\operatorname{trace} \bar{S})X$. Whence, by the Green's theorem, we deduce the Green's theorem

$$\int_M (\operatorname{trace} \bar{\nabla} X - 2(\operatorname{trace} \bar{S})X)\, dV = 0$$

for an arbitrary compact Riemann-Cartan manifold $(M, g, \bar{\nabla})$.

Remark. S. Goldberg, K. Yano and S. Bochner, and also Y. Kubo, N. Rani and N. Prakash (see [3], [13], [20], [26] and [38]) proved their "vanishing theorem" on compact oriented Riemann-Cartan manifolds under the condition that div $X =$ trace $\bar{\nabla} X$. In this case Green's theorem has the form $\int (\operatorname{trace} \bar{\nabla} X)\, dV = 0$. These Riemann-Cartan manifolds belong to the class $\Omega_1 \oplus \Omega_3$.

3. Scalar and total scalar curvatures of Riemann-Cartan manifolds

3.1. **Definitions of scalar and total scalar curvatures of the Riemann-Cartan manifold.** Let $(M, g, \bar{\nabla})$ be a Riemann-Cartan manifold with positive-define metric g and the curvature tensor $\bar{R}$. The covariant curvature tensor $\bar{R}^b$ is a smooth section of the tensor bundle $\Lambda^2 M \otimes \Lambda^2 M$, therefore *the scalar curvature* $\bar{s} = \bar{s}(x)$ of the Riemann–Cartan manifold $(M, g, \bar{\nabla})$ may be defined by the formula (see also [32])

$$\bar{s}(x) = \sum_{i,j=1}^{n} g(\bar{R}(e_i, e_j)e_j, e_i)$$

as an analogy to the scalar curvature $s(x) = \sum_{i,j=1}^{n} g(R(e_i, e_j)e_j, e_i)$ of the Riemannian manifold (M, g) for an arbitrary orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_x M$.

It is well known that the nonsymmetric tensor $\overline{\mathrm{Ric}}(X, Y)$ defined as the trace of the map $Z_x \to \bar{R}(Z_x, X_x)Y_x$ for any $Z_x, Y_x, X_x \in T_x M$ is a *Ricci tensor* (see [18]) of the nonsymmetric linear connection $\bar{\nabla}$. Therefore we can write the following identity $\bar{s}(x) = \sum_{i=1}^{n} \overline{\mathrm{Ric}}(e_i, e_i)$. In particular, for the Weitzenböck connection $\bar{\nabla}$ we have the following identity $\bar{s} = 0$ (see [1] and [36]).

The dependence between the scalar curvatures s and $\bar{s}$ is described in the following formula (see [32])

$$\bar{s} = s - \left\| {}^{(1)}T \right\|^2 - \frac{n-2}{2} \left\| {}^{(2)}T \right\|^2 + \frac{1}{2} \left\| {}^{(3)}T \right\|^2 - 2 \operatorname{div}(\operatorname{trace} T)^{\#}. \tag{3.1}$$

By the formulas (iv) and (v) of Sec. 2 we can rewrite the formula (3.1) in the form

$$\bar{s} = s - \left\| {}^{(1)}\bar{S} \right\|^2 - 2(n-2) \left\| {}^{(2)}\bar{S} \right\|^2 + 2 \left\| {}^{(3)}\bar{S} \right\|^2 - 4 \operatorname{div}(\operatorname{trace} \bar{S})^{\#}. \tag{3.2}$$

In particular, for the Weitzenböck connection $\bar{\nabla}$ we have the formula

$$s = \left\|{}^{(1)}\bar{S}\right\|^2 + 2(n-2)\left\|{}^{(2)}\bar{S}\right\|^2 - 2\left\|{}^{(3)}\bar{S}\right\|^2 + 4\,\mathrm{div}(\mathrm{trace}\,\bar{S})^{\#}.$$

Let $(M, g, \bar{\nabla})$ be a compact Riemann–Cartan manifold. We define its *complete scalar curvature* as the number $\bar{s}(M) = \int_M \bar{s}\,dV$ analogously to the *total scalar curvature* $s(M) = \int_M s\,dV$ of a Riemannian manifold. From the formula (3.1) we deduce the dependence between the total scalar curvatures $s(M)$ and $\bar{s}(M)$ in the following integral formula

$$\bar{s}(M) = s(M) - \int_M \left(\left\|{}^{(1)}\bar{S}\right\|^2 + 2(n-2)\left\|{}^{(2)}\bar{S}\right\|^2 - 2\left\|{}^{(3)}\bar{S}\right\|^2\right)dV.$$

In particular, for the Weitzenböck connection we have the integral formula

$$s(M) = \int_M \left(\left\|{}^{(1)}\bar{S}\right\|^2 + 2(n-2)\left\|{}^{(2)}\bar{S}\right\|^2 - 2\left\|{}^{(3)}\bar{S}\right\|^2\right)dV.$$

3.2. **Algebraic conditions on scalar and total scalar curvatures for some classes of Riemann-Cartan manifolds and vanishes theorems.** Firstly, consider a Riemann–Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_3 which is characterized by the conditions ${}^{(1)}\bar{S} = {}^{(2)}\bar{S} = 0$. For these conditions the identity (3.2) rewrite as $\bar{s} = s + 2\left\|{}^{(3)}\bar{S}\right\|^2$. Hence we have $\bar{s} \geq s$, where equality is possible only if $\bar{\nabla} = \nabla$. The following theorem holds (see also [32]).

Theorem 3.1. *The scalar curvatures $\bar{s}$ and s of the metric connection $\bar{\nabla}$ and of the Levi-Civita connection ∇ of an n-dimensional Riemann–Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_3 satisfy the inequality $\bar{s} \geq s$. The equality $\bar{s} = s$ is possible only if $\bar{\nabla} = \nabla$.*

Obviously that for any Riemann–Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_3 with Weitzenböck connection $\bar{\nabla}$ the following identity $s = -2\left\|{}^{(3)}\bar{S}\right\|^2$ holds. Therefore we can formulate the first corollary.

Corollary 3.1. *The torsion tensor $\bar{S}$ of the metric connection $\bar{\nabla}$ and scalar curvatures s of the Levi-Civita connection ∇ of an n-dimensional Riemann-Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_3 satisfy the identity $s = -2\,\|{}^{(3)}\bar{S}\|^2$. There exist no Weitzenböck connections of the class Ω_3 on a Riemannian manifold with $s > 0$.*

Knowing the definition of the scalar curvature $\bar{s}$ and taking account of the positive definiteness of the metric g, we can prove the following corollary (see also [32]).

Corollary 3.2. *On compact oriented Riemannian manifold (M, g) with positive semi-definite (resp. positive-definite) scalar curvature s, there is no nonsymmetric metric connection $\bar{\nabla}$ of class Ω_3 with negative-definite (resp. negative semi-definite) quadratic form $\overline{\mathrm{Ric}}(X, X)$ for the Ricci tensor $\overline{\mathrm{Ric}}$ of the connection $\bar{\nabla}$ and any smooth vector field X.*

Secondary, we consider a Riemann-Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_1. In this case the following theorem holds.

Theorem 3.2. *The scalar curvatures $\bar{s}$ and s of the metric connection $\bar{\nabla}$ and of the Levi-Civita connection ∇ of an n-dimensional Riemannian-Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_1 satisfy the inequality $\bar{s} \leq s$. The equality $\bar{s} = s$ is possible only if $\bar{\nabla} = \nabla$.*

For the proof we remark that the class Ω_1 characterized by the conditions ${}^{(2)}\bar{S} = {}^{(3)}\bar{S} = 0$ that is equal to $\bar{S}^b \in C^\infty\Lambda^3 M$. For these conditions the identity (3.2) may be rewritten as $\bar{s} = s - \left\| {}^{(1)}\bar{S} \right\|^2$. Hence we have $\bar{s} \leq s$, and equality is possible only if $\bar{\nabla} = \nabla$.

Obviously that for any Riemann-Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_1 with Weitzenböck connection $\bar{\nabla}$ the following identity $s = \left\| {}^{(1)}\bar{S} \right\|^2$ holds. Therefore we can formulate the third corollary.

Corollary 3.3. *The torsion tensor $\bar{S}$ of the metric connection $\bar{\nabla}$ and scalar curvatures s of the Levi-Civita connection ∇ of an n-dimensional Riemann-Cartan manifold $(M, g, \bar{\nabla})$ of the class Ω_1 satisfy the identity $s = \| {}^{(1)}\bar{S} \|^2$. Therefore there are no Weitzenböck connections of the class Ω_1 on a Riemannian manifold with $s < 0$.*

Using the definition of the scalar curvature $\bar{s}$ and taking account of the positive definiteness of the metric g, we can prove the following the forth corollary.

Corollary 3.4. *On compact oriented Riemannian manifold (M, g) with negative-semidefinite (resp. negative-definite) scalar curvature s, there is no non-symmetric metric connection $\bar{\nabla}$ of class Ω_1 with positive-definite (resp. positive-semidefinite) quadratic form $\overline{Ric}(X, X)$ for the Ricci tensor $\overline{Ric}$ of the connection $\bar{\nabla}$ and any smooth vector field X.*

Thirdly, consider a compact Riemann–Cartan manifold $(M, g, \bar{\nabla})$ of the class $\Omega_1 \oplus \Omega_2$, we have the following integral formula

$$\bar{s}(M) = s(M) - \int_M \left(\left\|{}^{(1)}\bar{S}\right\|^2 + 2(n-2)\left\|{}^{(2)}\bar{S}\right\|^2\right) dV.$$

For $n = 2$, equality is possible only if $\bar{S} = {}^{(3)}\bar{S}$. Then the following theorem is true (see also [32]).

Theorem 3.3. *The total scalar curvatures $s(M)$ and $\bar{s}(M)$ of Riemannian (M, g) compact manifold and a compact Riemann–Cartan manifold $(M, g, \bar{\nabla})$ of class $\Omega_1 \oplus \Omega_2$ are related by the inequality $\bar{s}(M) \leq s(M)$. For $\dim M \geq 3$, the equality is possible if the metric connection $\bar{\nabla}$ coincides with the Levi-Civita connection ∇ of the metric g, for $n = 2$, if $\bar{\nabla}$ is a semi-symmetric connection.*

Obviously that for any Riemann–Cartan manifold $(M, g, \bar{\nabla})$ with the Weitzenböck connection $\bar{\nabla}$ of the class $\Omega_1 \oplus \Omega_2$ the identity

$$s(M) = \int_M \left(\left\|{}^{(1)}\bar{S}\right\|^2 + 2(n-2)\left\|{}^{(2)}\bar{S}\right\|^2\right) dV$$

holds. Now we can formulate the corollary.

Corollary 3.5. *There are not Weitzenböck connections $\bar{\nabla}$ of the class $\Omega_1 \oplus \Omega_2$ on a compact Riemannian manifold with $s(M) \leq 0$.*

Regarding the definition of the scalar curvature $\bar{s}$ and taking account of the positive definiteness of the metric g, we can prove the following corollary (see also [32]).

Corollary 3.6. *On a compact Riemannian manifold (M, g) with negative semi-definite (resp. negative-definite) scalar curvature s, there is no nonsymmetric metric connection $\bar{\nabla}$ of class $\Omega_1 \oplus \Omega_2$ with positive-definite (resp. positive semi-definite) quadratic form $\overline{\mathrm{Ric}}(X, X)$ for the Ricci tensor $\overline{\mathrm{Ric}}$ of the connection $\bar{\nabla}$ and any smooth vector field X.*

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