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FOUR SERIES OF HYPERBOLIC SPACE GROUPS WITH SIMPLICIAL DOMAINS, AND THEIR SUPERGROUPS

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Abstract. Hyperbolic space groups are isometry groups, acting discontinuously on the hyperbolic 3-space with compact fundamental domain. One possibility to classify them is to look for fundamental domains of these groups.

Here are considered supergroups for four series of groups with simplicial fundamental domains. Considered simplices, denoted in [9] by T_{19}, T_{46}, T_{59}, belong to family F12, while T_{31} belongs to F27.

1. Introduction

Hyperbolic space groups are isometry groups, acting discontinuously on the hyperbolic 3-space with compact fundamental domain. One possibility to classify them is to look for fundamental domains of these groups. Face pairing identifications of a given polyhedron give us generators and relations for a space group by Poincaré Theorem [1], [3], [7].

The simplest fundamental domains are simplices and truncated simplices by polar planes of vertices when they lie out of the absolute. There are 64 combinatorially different face pairings of fundamental simplices [16], [6], furthermore 35 solid transitive non-fundamental simplex identifications [6]. I. K. Zhuk [16] has classified Euclidean and hyperbolic fundamental simplices of finite volume up to congruence. Some completing cases are discussed in [2], [5], [10], [11], [12], [13], [14], [15]. Algorithmic

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procedure is given by E. Molnár and I. Prok [5]. In [6], [8] and [9] the authors summarize all these results, arranging identified simplices into 32 families. Each of them is characterized by the so-called maximal series of simplex tilings. Besides spherical, Euclidean, hyperbolic realizations there exist also other metric realizations in 3-dimensional simply connected homogeneous Riemannian spaces, moreover, metrically non-realizable topological simplex tilings occur as well [4].

When vertices are out of the absolute, the simplex is not compact and then we truncate it with polar planes of the vertices. The new compact polyhedron obtained in that way, let us call it trunc-simplex, is fundamental domain of some larger group. It has new triangular faces whose pairing gives new generators. For simplicity, here we require that the new pairing generators keep the original simplicial face structure. Other possibilities will be discussed elsewhere. Dihedral angles around new edges are $\pi/2$. That means that there will be four congruent polyhedra around them in a new fundamental space filing. These investigations have been initiated by the author (see e.g. [14]).

Each identified simplex, considered in this paper, has two equivalence classes for edges with three edges in each. Edges in the same class haven't common vertex. There are 4 different face pairings: T_{19}, T_{46}, T_{59} in family F12 and T_{31} in family F27 to investigate in this paper to extend the series tabled in [9].

In Section 2 we recall Poincaré Theorem which provides a method to construct discontinuously acting ismometry groups. In Section 3 we discuss the supergroups with trunc-simplices as fundamental domains, for each simplex series separately (see Figures 1, 6, 8, 10). Since all considered simplices have the same inner symmetry, namely a half-turn about axis line h in Figure 5, this also gives a possibility to consider supergroups by this property. This interesting phenomenon occurs at the first three series, but not at T_{31}.

2. Construction of discontinuously acting isometry groups

Generators and relations for a space group G with a given polyhedron P (a simplex or a trunc-simplex in the considered cases) as a fundamental domain can be obtained by the Poincarè theorem. It is necessary to consider all face pairing identifications of such domains. Those will be isometries, which generate an isometry group G and induce subdivision of vertices and oriented edge segments of P into equivalence classes, such that an edge segment does not contain two G-equivalent points in its interior.

Face pairing identifications are isometries satisfying conditions (a)–(c). They generate an isometry group G of a space of constant curvature.

(a) For each face $f_{g^{-1}}$ of P there is another face f_g and identifying isometry g which maps $f_{g^{-1}}$ onto f_g and P onto P^g, the neighbour of P along f_g.
(b) The isometry g^{-1} maps the face f_g onto $f_{g^{-1}}$ and P onto $P^{g^{-1}}$, joining the simplex P along $f_{g^{-1}}$.
(c) Each edge segment e_1 from any equivalence class (defined below) is successively surrounded by polyhedra P, $P^{g_1^{-1}}$, $P^{g_2^{-1}g_1^{-1}}$, $\dots$, $P^{g_r^{-1}\cdots g_2^{-1}g_1^{-1}}$, which fill an angular region of measure $2\pi/\nu$, with a natural number ν. An equivalence class consisting of edge segments e_1, e_2, $\dots$, e_r with dihedral angles $\varepsilon(e_1)$, $\varepsilon(e_2)$, $\dots$, $\varepsilon(e_r)$, respectively, is defined as follows.

Let us consider an edge segment, say e_1, and choose one of the two faces denoted by $f_{g_1^{-1}}$ whose boundary contains e_1. The isometry g_1 maps e_1 and $f_{g_1^{-1}}$ onto e_2 and f_{g_1}, respectively. There exists exactly one other face $f_{g_2^{-1}}$ with e_2 on its boundary, furthermore the isometry g_2 mapping e_2 and $f_{g_2^{-1}}$ onto e_3 and f_{g_2}, respectively, and so on. We obtain a cycle of isometries $g_1, g_2, \dots, g_r$ according to the scheme

$$(2.1)\qquad \left(e_1, f_{g_1^{-1}}\right) \xrightarrow{g_1} (e_2, f_{g_1})\,; \left(e_2, f_{g_2^{-1}}\right) \xrightarrow{g_2} (e_3, f_{g_2})\,; \dots; \left(e_r, f_{g_r^{-1}}\right) \xrightarrow{g_r} (e_1, f_{g_r})$$

where the symbols are not necessarily distinct. More precisely, we have two essentially different cases for the scheme (1).

1: if a plane reflection $m_i = g_i$ occurs then $e_{i+1} = e_i$, and we turn back to e_1, then, say, e_{-1} comes. Furthermore, another plane reflection $m_{-j} = g_{-j}$ shall appear in the cycle. Then each edge segment comes two times in the scheme (1), and the cycle transformation is of the form

$$c = g_1 g_2 \dots g_r = \left(g_1 \dots g_{i-1} m_i g_{i-1}^{-1} g_1^{-1}\right) \left(g_{-1}^{-1} g_{-j+1}^{-1} m_{-j} g_{-j+1} g_{-1}\right)$$

2: there is no plane reflection in the cycle; this will be the simpler case. (In dimension 3 we have 5 subcases for the edges at all [3]).

In other words the segment e_1 is successively surrounded by polyhedra

$$P, P^{g_1^{-1}}, P^{g_2^{-1}g_1^{-1}}, \dots, P^{g_r^{-1}\cdots g_2^{-1}g_1^{-1}}$$

which fill an angular region of measure $2\pi/\nu$. In the above case 1. the following holds

$$\varepsilon(e_1) + \cdots + \varepsilon(e_i) + \varepsilon(e_{-1}) + \cdots + \varepsilon(e_{-1+j}) = \pi/\nu. \tag{2.2}$$

In case 2. we have

$$\varepsilon(e_1) + \cdots + \varepsilon(e_r) = 2\pi/\nu. \tag{2.3}$$

Finally, the cycle transformation $c = g_1 g_2 \dots g_r$ belonging to the edge segment class $\{e_1\}$ is a rotation, say, of order ν. Thus we have the cycle relation in both cases

$$(g_1 g_2 \dots g_r)^\nu = 1. \tag{2.4}$$

Throughout in this paper we shall apply the specified Poincaré theorem:

Theorem 2.1. *Let P be a polyhedron in a space S^3 of constant curvature and G be the group generated by the face identifications, satisfying conditions (a)–(c). Then G is a discontinuously acting group on S^3, P is a fundamental domain for G and the cycle relations of type* (2.4) *for every equivalence class of edge segments form a complete set of relations for G, if we also add the relations $g_i^2 = 1$ to the occasional involutive generators $g_i = g_i^{-1}$.*

3. Isometry groups of simplices and their supergroups

3.1. SIMPLEX T_{19}

Face pairing isometries for simplex $T_{19}\,(6a, 6b)$ (Figure 1) are

$$r_0 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_3 & A_2 & A_1 \end{pmatrix}; \quad r_1 : \begin{pmatrix} A_0 & A_2 & A_3 \\ A_2 & A_0 & A_3 \end{pmatrix}; \quad r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_3 & A_1 & A_0 \end{pmatrix}; \quad r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_2 & A_1 \end{pmatrix}.$$

Relations for the isometry group are obtained by Theorem 2.1 and the presentation is

$$\Gamma(T_{19}, 6a, 6b) = (r_0, r_1, r_2, r_3 - r_0^2 = r_1^2 = r_2^2 = r_3^2 = (r_0 r_1 r_2 r_1 r_0 r_3)^a = (r_3 r_2 r_0 r_2 r_3 r_1)^b = 1; a, b \in N).$$

Considering vertex figures on a symbolic 2-dimensional surface (plane) around the vertices, we can glue a fundamental domain for the stabilizer subgroup, e.g. $\Gamma(A_2)$ of vertex A_2. Transformation r_1 maps vertex A_2 onto A_0 and T_{A_2} onto $T_{A_0}^{r_1}$. That means that T_{A_2} and $T_{A_0}^{r_1}$ have a joint edge corresponding to the joint face f_{r_1} of simplex T. Similarly, vertex figures T_{A_2} and $T_{A_1}^{r_3}$ have joint edge corresponding to f_{r_3}, and $T_{A_1}^{r_3}$ and $T_{A_3}^{r_0 r_3}$ to $(f_{r_0})^{r_3}$. One fundamental domain for $\Gamma(A_2)$ (Figure 2) is

$$P_{A_2} := T_{A_0}^{r_1} \cup T_{A_2} \cup T_{A_1}^{r_3} \cup T_{A_3}^{r_0 r_3}$$

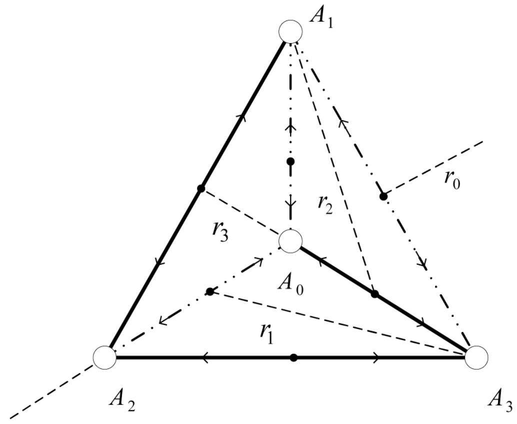

FIGURE 1. The simplex T_{19}

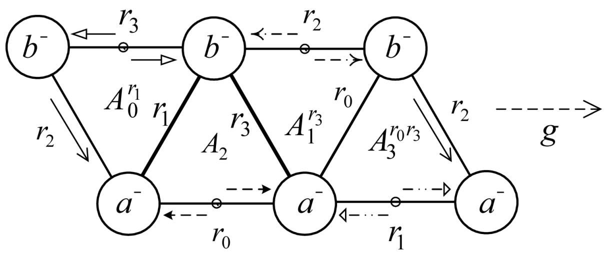

FIGURE 2. The fundamental domain P_{A_2} for Γ_{A_2}

and the generators for $\Gamma(A_2)$, obtained from P_{A_2}, are

$$r_3r_0r_2r_1 : (f_{r_2})^{r_0r_3} \to (f_{r_2})^{r_1}\,; \quad r_0 : f_{r_0} \to f_{r_0}; \quad r_1r_3r_1 : (f_{r_3})^{r_1} \to (f_{r_3})^{r_1}\,;$$

$$r_3r_2r_3 : (f_{r_2})^{r_3} \to (f_{r_2})^{r_3}\,; \qquad (r_3r_0)r_1(r_0r_3) : (f_{r_1})^{r_0r_3} \to (f_{r_1})^{r_0r_3}\,.$$

In the diagram for P_{A_2} the minus sign in notations a^-, b^- means that edges in these classes are directed to the considered vertex, (the plus sign in diagram means the opposite direction).

When parameters a, b are large enough, namely $1/a + 1/b < 2$, by angle sum criterion for P_{A_2}, then simplex T is hyperbolic with the vertices out of the absolute [9]. Then it is possible to truncate the simplex by polar planes of these vertices. In such a way we get a compact trunc-simplex (with 8 faces) denoted by $O_{19}(6a, 6b)$. If we equip O_{19} with additional face pairing isometries, it will be a fundamental domain for a group $\Gamma_j(O_{19}, 6a, 6b)$ which will be a supergroup of $\Gamma(T_{19}, 6a, 6b)$. We require, also later on, that the new generators keep the original simplex face structure. A trivial group extension with plane reflections $\overline{m}_i$, $i=0,1,2,3$, in polar planes of the outer vertices A_i is always possible (Figure 3). Then the new group, by Theorem 2.1 is

$$\Gamma_1(O_{19},6a,6b)=(r_0,r_1,r_2,r_3,\overline{m}_0,\overline{m}_1,\overline{m}_2,\overline{m}_3-r_0^2=r_1^2=r_2^2=r_3^2=$$
$$\overline{m}_0^2=\overline{m}_1^2=\overline{m}_2^2=\overline{m}_3^2=(r_0r_1r_2r_1r_0r_3)^a=(r_3r_2r_0r_2r_3r_1)^b=$$
$$\overline{m}_0r_3\overline{m}_0r_3=\overline{m}_1r_2\overline{m}_1r_2=\overline{m}_2r_0\overline{m}_2r_0=\overline{m}_3r_1\overline{m}_3r_1=\overline{m}_0r_2\overline{m}_3r_2=$$
$$\overline{m}_1r_3\overline{m}_2r_3=\overline{m}_0r_1\overline{m}_2r_1=\overline{m}_1r_0\overline{m}_3r_0=1).$$

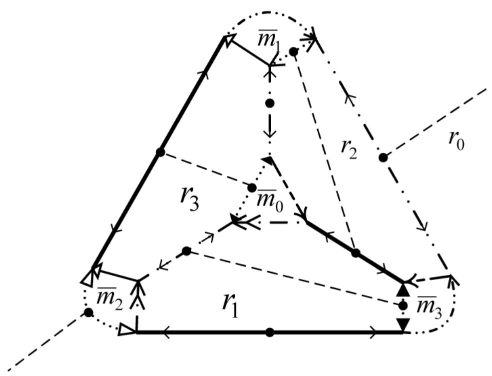

FIGURE 3. The trunc-simplex O_{19}^1 with trivial group extension

There is a further possibility to equip the new triangular faces with face pairing isometries (Figure 4). New additional face pairings of O_{19} have to satisfy the following criteria. Polar plane of A_2 and so stabilizer $\Gamma(A_2)$ will be invariant under these new transformations, fixing A_2, and exchanging the half spaces obtained by the polar plane. Thus, fundamental domain P_{A_2} is divided into two parts, and the new stabilizer of the polar plane will be a supergroup for $\Gamma(A_2)$, namely of index two. Inner symmetries of the P_{A_2}-tiling give us the idea how to introduce a new generators. Let g be the glide reflection as a composition of the translation in the plane of the vertex figure with a reflection in this plane. Then g maps the vertex figure T_{A_2} onto $T_{A_3}^{r_0r_3}$ and $T_{A_3}^{r_0r_3}$ onto $T_{A_2}^{r_1r_2r_0r_3}$, equivalent to T_{A_2}. Then g also maps $T_{A_0}^{r_1}$ onto $T_{A_1}^{r_3}$ and $T_{A_1}^{r_3}$ onto $T_{A_0}^{r_2r_0r_3}$, equivalent to $T_{A_0}^{r_1}$. In that case the new generators for $\Gamma_2(O_{19},6a,6b)$ will be g_1 and $g_2=r_1g_1r_0$ in Figure 4, while the new group, by Theorem 2.1 is

$$\Gamma_2(O_{19},6a,6b)=(r_0,r_1,r_2,r_3,g_1,g_2-r_0^2=r_1^2=r_2^2=r_3^2=(r_0r_1r_2r_1r_0r_3)^a=$$
$$(r_3r_2r_0r_2r_3r_1)^b=r_3g_1r_2g_1^{-1}=g_1r_3g_2r_2=g_1r_0g_2^{-1}r_1=r_0g_2r_1g_2^{-1}=1).$$

The P_{A_2}-tiling in the polar plane of A_2 do not allow other identifications on the truncated simplex O_{19}.

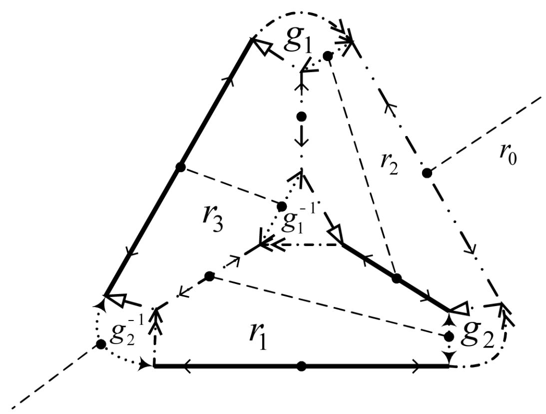

FIGURE 4. The trunc-simplex O_{19}^2 with non-trivial group extension

Fundamental domains T_{19} and O_{19}^j ($j = 1, 2$) above, allow to divide them to smaller polyhedra, equipped with face pairing identifications. Namely, there is a half-turn h

$$h : \begin{pmatrix} A_0 & A_1 & A_2 & A_3 \\ A_1 & A_0 & A_3 & A_2 \end{pmatrix}$$

leaving invariant the tessellations of space with T_{19} or O_{19}^j, so groups $\Gamma(T_{19}, 6a, 6b)$ and $\Gamma_j(O_{19}, 6a, 6b)$ are not maximal. The authomorphism groups ${}_2^2\Gamma_6(3u, 3v)$ of their tilings ([8], [9]) have domains which are fundamental polyhedra of piecewise linear bent faces. That domains are obtained by identifying equivalent points, under symmetry h, of simplex T_{19} (Figure 5), and consequently also each trunc-simplex O_{19}^j above ($j = 1, 2$).

Since $r_3 = hr_2h$ and $r_1 = hr_0h$, presented for $a \neq b$, maximal groups are now (with $u = 2a$ and $v = 2b$ for the rotational parameters) by

$$\begin{aligned} {}_2^2\Gamma_6(3u, 3v) =&(h, r_0, r_2 - h^2 = r_0^2 = r_2^2 = (r_0hr_0hr_2h)^u = \\ &(r_2hr_2r_0)^v = 1; u = 2a, v = 2b) \end{aligned}$$

and

$$\Gamma(Q, 3u, 3v) = (h, r_0, r_2, \overline{m}_1, \overline{m}_2 - h^2 = r_0^2 = r_2^2 = \overline{m}_1^2 = \overline{m}_2^2 = (r_0hr_0hr_2h)^u =$$

$$(r_2hr_2r_0)^v = \overline{m}_1r_2\overline{m}_1r_2 = \overline{m}_2r_0\overline{m}_2r_0 = \overline{m}_1r_2\overline{m}_2r_2 = \overline{m}_1r_0\overline{m}_2r_0 = 1; u = 2a, v = 2b).$$

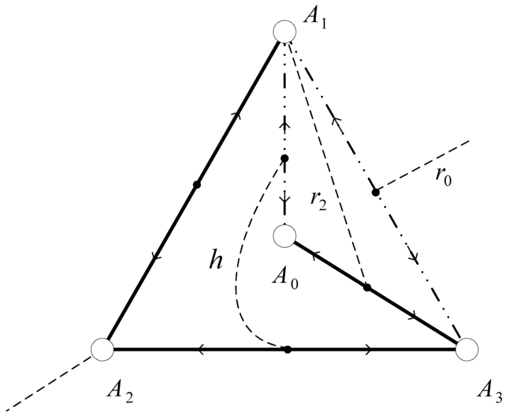

FIGURE 5. The fundamental domain of supergroup ${}^2_2\Gamma_6(3u, 3v)$

If $a = b$ then simplex T and trunc-simplex O^j have more symmetries. Then the maximal supergroup for $\Gamma(T_{19}, 6a, 6b)$ is a Coxeter group, by [9], while the maximal supergroup for $\Gamma_j(O_{19}, 6a, 6b)$ might have only the trivial extension, so it is also a Coxeter group.

3.2. SIMPLEX T_{46}

For $T_{46}(6a, 3b)$, the face pairing isometries are (Figure 6):

$$r_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_3 & A_1 & A_0 \end{pmatrix}; \quad r_3 : \begin{pmatrix} A_0 & A_1 & A_2 \\ A_0 & A_2 & A_1 \end{pmatrix}; \quad s : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_3 & A_0 \end{pmatrix},$$

and the tiling group is

$$\Gamma(T_{46}, 6a, 3b) = (r_2, r_3, s - r_2^2 = r_3^2 = (s^2 r_2 s^{-2} r_3)^a = (r_2 s r_3)^b = 1; a, b \in N).$$

One fundamental domain for the stabilizer group $\Gamma(A_2)$ of the vertex A_2 (Figure 6) is

$$P_{A_2} := T_{A_0}^{r_2 s^{-1}} \cup T_{A_3}^{s^{-1}} \cup T_{A_2} \cup T_{A_1}^{r_3}$$

and the generators are then

$$s r_2 r_3 r_2 s^{-1} : (f_{r_3})^{r_2 s^{-1}} \to (f_{r_3})^{r_2 s^{-1}}; \quad s^2 r_2 s^{-1} : (f_s^{-1})^{s^{-1}} \to (f_s)^{r_2 s^{-1}};$$

$$r_3 s : (f_{s^{-1}})^{r_3} \to f_s; \quad r_3 r_2 r_3 : (f_{r_2})^{r_3} \to (f_{r_2})^{r_3}.$$

The stabilizer $\Gamma(A_2)$ of P_{A_2} above is hyperbolic iff (again by the angle sum criterion for P_{A_2}) $2/b + 1/a < 2$. Then truncating the simplex by polar planes of the vertices,

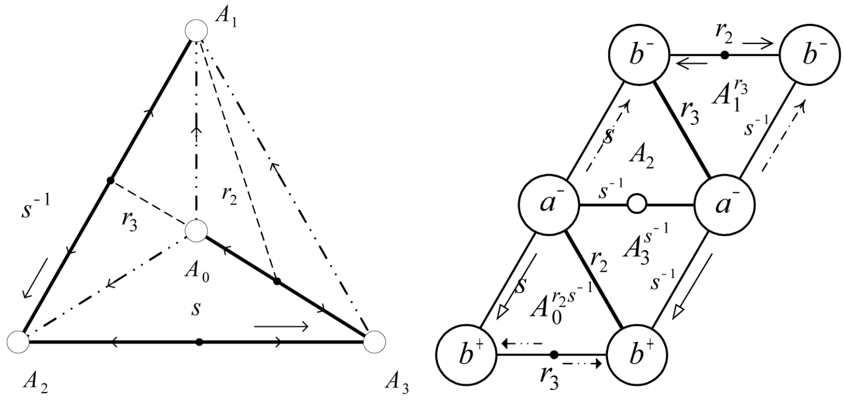

FIGURE 6. The simplex T_{46} and the fundamental domain P_{A_2}

a new trunc-simplex O_{46} may have plane reflections as face pairing isometries of the new faces. In this case the new group is (Figure 7)

$$\Gamma_1(O_{46}, 6a, 3b) = (r_2, r_3, s, \overline{m}_0, \overline{m}_1, \overline{m}_2, \overline{m}_3 - r_2^2 = r_3^2 = \overline{m}_0^2 = \overline{m}_1^2 = \overline{m}_2^2 = \overline{m}_3^2 = (s^2 r_2 s^{-2} r_3)^a = (r_2 s r_3)^b = \overline{m}_0 r_3 \overline{m}_0 r_3 = \overline{m}_1 r_2 \overline{m}_1 r_2 = \overline{m}_0 r_2 \overline{m}_3 r_2 = \overline{m}_1 r_3 \overline{m}_2 r_3 = \overline{m}_2 s \overline{m}_3 s^{-1} = \overline{m}_3 s \overline{m}_0 s^{-1} = \overline{m}_1 s \overline{m}_2 s^{-1} = 1).$$

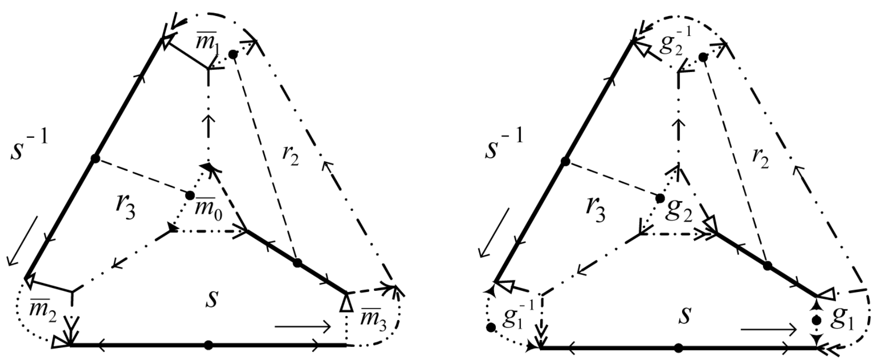

FIGURE 7. The trunc-simplex O_{46}

Other possibility, by symmetries of the fundamental domain P_{A_2} is the group extended by the point reflection z, indicated in Figure 6. This point reflection reflection z (say) maps the triangle of A_2 to that of $A_3^{s^{-1}}$ and triangle of $A_1^{r_3}$ to that of $A_0^{r_2 s^{-1}}$ in P_{A_2} (Figure 6). Thus, the above z induces new generators g_1 and g_2 as glide reflections, pairing the truncations at A_2, A_3 and those at A_1, A_0, respectively.

$$\Gamma_2(O_{46}, 6a, 3b) = (r_2, r_3, s, g_1, g_2 - r_2^2 = r_3^2 = (s^2 r_2 s^{-2} r_3)^a = (r_2 s r_3)^b = r_2 g_2 r_3 g_2^{-1} = g_2 r_2 g_1^{-1} r_3 = s g_1 s g_2^{-1} = g_1 s^{-1} g_1 s^{-1} = 1).$$

If r_0 and h are similarly introduced, as in the previous section, so that $r_3 = h r_2 h$ and $s = r_0 h$ hold. Then the maximal group ${}^2_2\Gamma_6(3u, 3v)$, now with $u = 2a$, $v = b$, will be supergroup of $\Gamma(T_{46}, 6a, 3b)$, and $\Gamma(Q, 3u, 3v)$ extends $\Gamma_j(O_{46}, 6a, 3b)$ $(j = 1, 2)$ as well.

3.3. SIMPLEX T_{59}

In the case of the simplex $T_{59}(3a, 3b)$ the face pairing identifications are (Figure 8)

$$s_1 : \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_3 & A_0 \end{pmatrix}; \qquad s_2 : \begin{pmatrix} A_0 & A_1 & A_3 \\ A_2 & A_0 & A_1 \end{pmatrix}$$

and the presentation of the group is

$$\Gamma(T_{59}, 3a, 3b) = (s_1, s_2 - (s_1^2 s_2)^a = (s_2^2 s_1^{-1})^b = 1; a, b \in N).$$

The stabilizer group $\Gamma(A_0)$ has fundamental domain (Figure 8)

$$P_{A_0} := T_{A_3}^{s_2^2} \cup T_{A_1}^{s_2} \cup T_{A_0} \cup T_{A_2}^{s_2^{-1}}$$

and the generators

$$s_2^{-2} s_1 : (f_{s_1^{-1}})^{s_2^2} \to f_{s_1}; \quad s_2^{-1} s_1 s_2^{-1} : (f_{s_1^{-1}})^{s_2} \to (f_{s_1})^{s_2^{-1}}; \quad s_2 s_1 s_2^2 : (f_{s_1^{-1}})^{s_2^{-1}} \to (f_{s_1})^{s_2^2}.$$

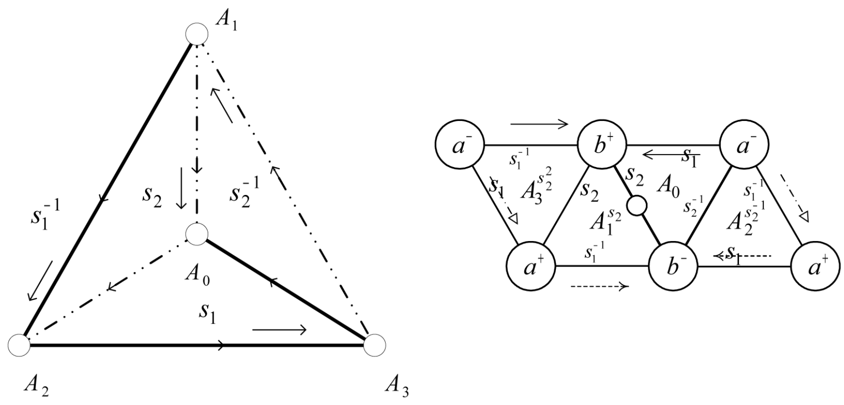

FIGURE 8. The simplex T_{59} and the fundamental domain P_{A_0}

There are two possibilities for the isometry group with trunc-simplex O_{59} as a fundamental domain, iff $1/a + 1/b < 1$. In the trivial case, group is (Figure 9)

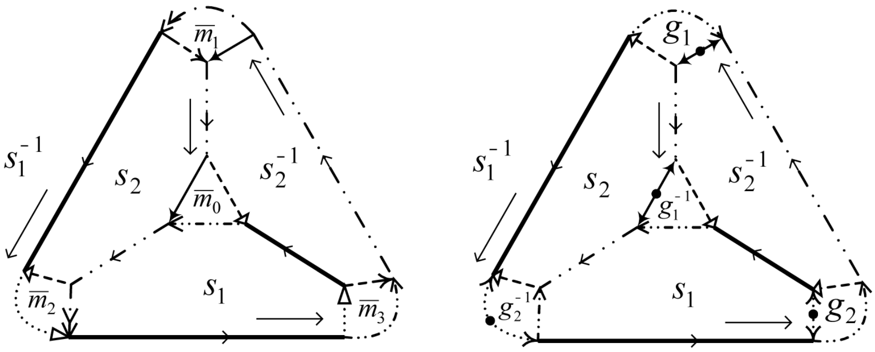

FIGURE 9. The trunc-simplex O_{59}

$$\begin{aligned}\Gamma_1(O_{59},3a,3b) =&(s_1,s_2,\overline{m}_0,\overline{m}_1,\overline{m}_2,\overline{m}_3 - \overline{m}_0^2=\overline{m}_1^2=\overline{m}_2^2=\overline{m}_3^2=\\ &= (s_1^2s_2)^a=(s_2^2s_1^{-1})^b=\overline{m}_2s_1\overline{m}_3s_1^{-1}=\overline{m}_3s_1\overline{m}_0s_1^{-1}=\\ &\overline{m}_1s_1\overline{m}_2s_1^{-1}=\overline{m}_3s_2\overline{m}_1s_2^{-1}=\overline{m}_1s_2\overline{m}_0s_2^{-1}=\overline{m}_0s_2\overline{m}_2s_2^{-1}=1).\end{aligned}$$

Taking g_1 and $g_2 = s_2^{-1}g_1s_2^{-1}$ as a new generators, other possibility for the group is

$$\begin{aligned}\Gamma_2(O_{59},3a,3b) =&(s_1,s_2,g_1,g_2 - (s_1^2s_2)^a=(s_2^2s_1^{-1})^b=g_1s_1g_2s_1=g_1s_2g_1s_2=\\ &s_2g_2s_2g_1^{-1}=g_2s_1^{-1}g_2s_1^{-1}=1).\end{aligned}$$

Since, it is possible to express the face pairing isometries s_1 and s_2 of T_{59} by h, r_0, r_2: $s_1 = r_0h$ and $s_2 = r_2h$, the groups ${}^2_2\Gamma_6(3u,3v)$ and $\Gamma(Q,3u,3v)$ are supergroups of the groups $\Gamma(T_{59},3a,3b)$ and $\Gamma_j(O_{59},3a,3b)$, $(u=a, v=b)$.

3.4. SIMPLEX T_{31}

The face pairings identifications for the simplex $T_{31}(6a,12b)$ are (Figure 10)

$$m:\begin{pmatrix}A_1 & A_2 & A_3\\ A_1 & A_2 & A_3\end{pmatrix};\quad r:\begin{pmatrix}A_0 & A_2 & A_3\\ A_2 & A_0 & A_3\end{pmatrix};\quad s:\begin{pmatrix}A_0 & A_1 & A_2\\ A_1 & A_3 & A_0\end{pmatrix}.$$

The group presentation is

$$\begin{aligned}\Gamma(T_{31},6a,12b) =&(m,r,s - r^2=m^2=(rmrs^{-1}ms)^a=\\ &(rs^2ms^{-2}rs^2ms^{-2})^b=1; a\geq 1, b\geq 1).\end{aligned}$$

For the stabilizer group $\Gamma(A_1)$ one of the fundamental domains is (Figure 10)

$$P_{A_1} := T_{A_2}^{s^2}\cup T_{A_0}^{s}\cup T_{A_1}\cup T_{A_3}^{s^{-1}}$$

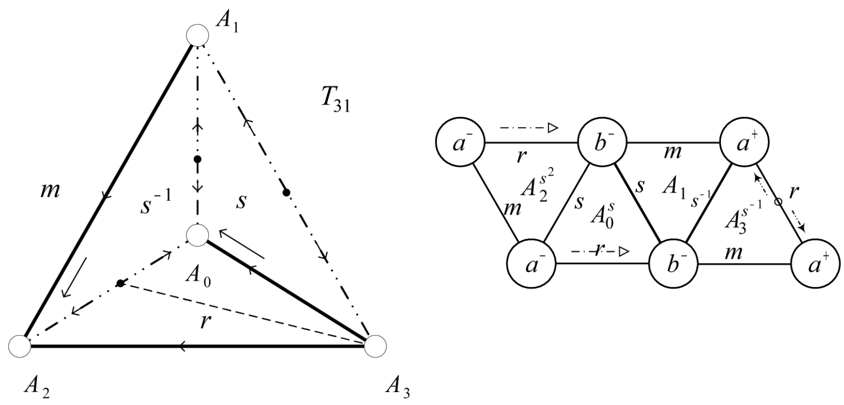

FIGURE 10. The simplex T_{31} and the fundamental domain P_{A_1}

with generators

$$srs^{-1} : (f_r)^{s^{-1}} \to (f_r)^{s^{-1}}; \qquad s^{-1}rs : (f_r)^{s^2} \to (f_r)^{s}.$$

After truncating the simplex by the polar planes of the vertices, iff $1/b + 1/a < 4$ trunc-simplex O_{31} may have **only** trivial group extension (Figure 11)

$$\begin{aligned}\Gamma(O_{31}, 6a, 12b) =&(m, r, s - r^2 = m^2 = (rmrs^{-1}ms)^a = (rs^2ms^{-2}rs^2ms^{-2})^b = \\ &\overline{m}_3 r \overline{m}_3 r = \overline{m}_0 r \overline{m}_2 r = \overline{m}_1 m \overline{m}_1 m = \overline{m}_2 m \overline{m}_2 m = \overline{m}_3 m \overline{m}_3 m = \\ &\overline{m}_1 s \overline{m}_3 s^{-1} = \overline{m}_2 s \overline{m}_0 s^{-1} = \overline{m}_0 s \overline{m}_1 s^{-1} = 1; a \geq 1, b \geq 1).\end{aligned}$$

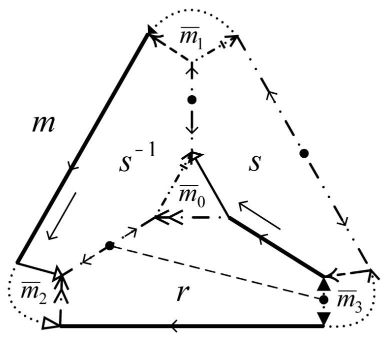

FIGURE 11. The trunc-simplex O_{31}^1

It is not possible to extend generators of T_{31} by h, since then a new reflection plane on halfturn axis r would yield $a = b$ and we got the richer family F.1.

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